Miyaoka-Yau inequality for hyperplane arrangements

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BIMSA AG Seminar

23/10/2025

Plan

- Main result: $Q \leq 0$ on C
- Proof
- Examples

The (Hirzebruch) quadratic form Q

- Let $\mathcal{H} = \{H_1, \ldots, H_k\}$ be a hyperplane arrangement, i.e., a finite set of pairwise distinct complex hyperplanes $H_i \subset \mathbb{CP}^n$
- The **multiplicity** of a linear subspace $L \subset \mathbb{CP}^n$ is

$$m(L) := \left|\{H_i \in \mathcal{H} \mid H_i \supset L\}\right|$$

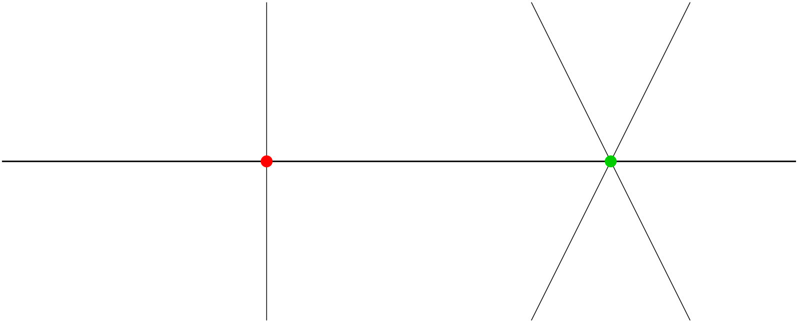


Figure 1: Multiplicity 2 (red) and 3 (green).

The Hirzebruch quadratic form Q

- $\mathcal{H} = \{H_1, \ldots, H_k\}$ hyperplane arrangement in $\mathbb{CP}^n$
- σ_i = number of codimension 2 subspaces $L \subset H_i$ with $m(L) \geq 3$
- The **Hirzebruch quadratic form** of $\mathcal{H}$ is the homogeneous degree 2 polynomial on $\mathbb{R}^k$ given by

$$Q(a_1, \ldots, a_k) = \sum_{i,j=1}^{k} Q_{ij} a_i a_j$$

$$Q_{ij} = \begin{cases} -(n+1)\sigma_i + 2n & \text{if } i = j \\ -2 & \text{if } i \neq j \text{ and } m\big(H_i \cap H_j\big) = 2 \\ \ n-1 & \text{if } i \neq j \text{ and } m\big(H_i \cap H_j\big) \geq 3 \end{cases}$$

Line arrangements (Hirzebruch, 1983-85)

Let σ_i be the number of points p with $r_p \geqslant 3$ lying on the i-th line of the given arrangement of k lines in the plane. We consider the $(k \times k)$-symmetric matrix A with

$$A_{ij} = \begin{cases} 3\sigma_i - 4 & (i = j), \\ 2 & (i \neq j, \quad p \in L_i \cap L_j \text{ with } r_p = 2), \\ -1 & (i \neq j, \quad p \in L_i \cap L_j \text{ with } r_p \geqslant 3). \end{cases} \tag{3}$$

With the k lines we associate real variables x_i and let x be the column vector $(x_1, \ldots, x_k)$. With the s points p_j with $r_{p_j} \geqslant 3$ we associate real variables y_j. For each point p_j with $r_{p_j} \geqslant 3$ we consider the linear form

$$P_j(x, y) = 2y_j + \sum_{p_j \in L_i} x_i, \quad \text{where} \quad y = (y_1, \ldots, y_s).$$

Höfer's formula. *For the algebraic surface Y (a good covering of S of degree d with respect to $L_1, \ldots, L_k, E_1, \ldots, E_s$ and the given branching numbers $n_1, \ldots, n_k, m_1, \ldots, m_s$) we have*

$$(3c_2(Y) - c_1^2(Y))/d = \frac{1}{4}\left(x^t A x + \sum_{j=1}^{s} P_j(x, y)^2\right), \tag{4}$$

where $x_i = 1 - \frac{1}{n_i}$ *and* $y_j = -1 - \frac{1}{m_j}$.

The (semi-stable) cone C

- A **basis** of $\mathcal{H} = \{H_1, \ldots, H_k\}$ is a subset $\mathcal{B} \subset \mathcal{H}$ consisting of $n+1$ linearly independent hyperplanes
- The indicator vector of $\mathcal{B}$ is the 1/0 vector

$$\mathbf{e}_{\mathcal{B}} = \sum_{i \,|\, H_i \in \mathcal{B}} \mathbf{e}_i$$

  where $\mathbf{e}_1, \ldots, \mathbf{e}_k$ are the standard basis vectors of $\mathbb{R}^k$
- The **matroid polytope** is the convex hull of the vectors $\mathbf{e}_{\mathcal{B}}$

$$P = \operatorname{conv}\{\, \mathbf{e}_{\mathcal{B}} \mid \mathcal{B} \text{ is a basis of } \mathcal{H} \,\}$$

Note: P is contained in the $(k-1)$-simplex $\Delta \subset \mathbb{R}^k$ with

$$\Delta = \left\{\, (a_1, \ldots, a_k) \in \mathbb{R}^k \,|\, a_i \geq 0,\ \sum_i a_i = n+1 \,\right\}$$

The semistable cone C

- The **semistable cone** is the cone over the matroid polytope

$$C = \mathbb{R}_{\geq 0} \cdot P = \text{cone}\,\{\, \mathbf{e}_{\mathcal{B}} \mid \mathcal{B} \text{ is a basis of } \mathcal{H} \,\}$$

  It is a convex polyhedral cone contained in the octant $\left(\mathbb{R}_{\geq 0}\right)^k$

- The **stable cone** is the interior of $C \subset \mathbb{R}^k$

$$C^\circ = \text{int}(C)$$

- $\mathcal{H}$ is *essential* and *irreducible* $\iff \dim P = k - 1$
  $\iff C^\circ$ is non empty

The stable cone C°

- Let $\mathcal{L}$ be the finite set of non-empty and proper linear subspaces $L \subset \mathbb{CP}^n$ obtained by intersecting members of $\mathcal{H}$
- Let $\mathbf{a} = (a_1, \ldots, a_k) \in (\mathbb{R}_{>0})^k$ then $\mathbf{a} \in C^\circ \iff \forall L \in \mathcal{L}:$

$$\sum_{i \mid H_i \supset L} a_i < \frac{\operatorname{codim} L}{n+1} \cdot \sum_{i=1}^{k} a_i$$

Relation to Geometric Invariant Theory

- Standard embedding $\mathbb{CP}^n \subset \mathfrak{su}(n+1)^*$ as a coadjoint orbit, e.g., if $n = 1$ then $\mathbb{CP}^1 = S^2 \subset \mathbb{R}^3$
- Let $p_i \in (\mathbb{CP}^n)^*$ be the annihilator of H_i
- $(a_1, \ldots, a_k) \in C^\circ \iff \exists\ F \in SL(n+1, \mathbb{C})$ such that the centre of mass of the points $F(p_i)$ with weights a_i is $0 \in \mathfrak{su}(n+1)^*$

Main Result

Theorem (Miyaoka-Yau inequality, dB-Panov 2024)

The Hirzebruch quadratic form is non-positive on the semistable cone:

$$C \subset \{Q \leq 0\}$$

- If $n = 2$ this follows from Panov's *Polyhedral Kähler Manifolds*, Geometry & Topology, 2009
- **Conjecture:** if $\mathbf{a} = (a_1, \ldots, a_k) \in C^\circ$ is such that $Q(\mathbf{a}) = 0$ and $a_i \in (0, 1)$. Then there is a Kähler metric on $\mathbb{CP}^n$ of constant holomorphic sectional curvature with cone angles $2\pi(1 - a_i)$ along the hyperplanes $H_i \in \mathcal{H}$
    - If $\mathcal{H}$ is a complex reflection arrangement then the metrics have been constructed by Couwenberg-Heckman-Looijenga (IHÉS, 2005)
    - dB-Panov, *Polyhedral Kähler metrics on* $\mathbb{CP}^n$, arXiv:2510.17447

Proof: klt and CY arrangements

- We can assume that $\mathcal{H}$ is essential and irreducible
- It is enough to show that if $\mathbf{a} \in C^\circ$ then $Q(\mathbf{a}) \leq 0$
- after scaling $\mathbf{a} \mapsto \lambda \cdot \mathbf{a}$ we can assume that

$$\sum_i a_i = n + 1$$

- $\mathbf{a} \in C^\circ$ is equivalent to

$$\forall L \in \mathcal{L} : \quad \sum_{i \,|\, H_i \supset L} a_i < \operatorname{codim} L$$

  Taking $L = H_i$ we have $a_i \in (0, 1)$
- the pair $\left(\mathbb{CP}^n, \sum a_i H_i\right)$ is klt and Calabi-Yau (CY)

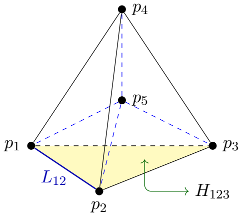
Proof: the resolution

- Logarithmic resolution
$$X \xrightarrow{\pi} \mathbb{CP}^n$$
with $D = \pi^{-1}(\mathcal{H})$ a simple normal crossing divisor
- X is the *minimal De Concini-Procesi wonderful model* of $\mathcal{H}$
- X is obtained by blowing up the *irreducible subspaces* $\mathcal{L}_{\mathrm{irr}} \subset \mathcal{L}$ in increasing order of dimension
- The irreducible components of D are in bijective correspondence with non-empty and proper irreducible subspaces $L \in \mathcal{L}_{\mathrm{irr}}$
$$D = \bigcup_{L \in \mathcal{L}_{\mathrm{irr}}} D_L$$
where D_L is the unique irreducible component of D such that
$$\pi(D_L) = L$$

Proof: the resolution (example)

Let $p_1, \ldots, p_5 \in \mathbb{CP}^3$ be five points in general linear position

$$\mathcal{H} \; = 10 \text{ planes}$$



- **Step 1:** $X_1 \xrightarrow{\sigma_1} \mathbb{CP}^3$ is the blowup at the five points p_i
- **Step 2:** $X \xrightarrow{\sigma_2} X_1$ is the blowup at the 10 (disjoint) proper transforms of the lines L_{ij}

$$\pi^{-1}(\mathcal{H}) = \underbrace{\left(\bigcup_H D_H\right)}_{10 \text{ divisors } \cong \mathrm{Bl}_4 \mathbb{P}^2} \bigcup \underbrace{\left(\bigcup_L D_L\right)}_{10 \text{ divisors } \cong \mathbb{P}^1 \times \mathbb{P}^1} \bigcup \underbrace{\left(\bigcup_p D_p\right)}_{5 \text{ divisors } \cong \mathrm{Bl}_4 \mathbb{P}^2}$$

Remark: $X = \overline{M_{0,6}}$

Proof: the parabolic bundle

Parabolic bundle $\mathcal{E}_*$ on (X, D) defined by:

- vector bundle $\mathcal{E} = \pi^*(T\mathbb{CP}^n)$
- weights a_L for $L \in \mathcal{L}_{\text{irr}}$ given by

$$a_L = (\operatorname{codim} L)^{-1} \sum_{i \,|\, H_i \supset L} a_i$$

  klt $\iff a_L \in (0, 1)$

- increasing filtrations of $\mathcal{E}|_{D_L}$ by vector subbundles

$$F_a^L = \begin{cases} \pi^*(TL) & \text{if } 0 \leq a < a_L \\ \mathcal{E}|_{D_L} & \text{if } a_L \leq a < 1 \end{cases}$$

  CY $\iff$ $\text{par-c}_1(\mathcal{E}_*) = 0$

Proof: the stability theorem

- Fix positive integers b_L for L in $\mathcal{L}_{\mathrm{irr}}^{\circ} = \mathcal{L}_{\mathrm{irr}} \setminus \mathcal{H}$ such that

  $$P_N = N \cdot \pi^*\big(\mathcal{O}_{\mathbb{P}^n}(1)\big) - \sum_{L \in \mathcal{L}_{\mathrm{irr}}^{\circ}} b_L \cdot D_L$$

  is an ample line bundle on X for all $N \gg 1$

- **Stability Theorem.** If $\mathcal{V} \subset \mathcal{E}$ is a proper *saturated subsheaf* then

  $$\text{par-c}_1(\mathcal{V}_*) \cdot c_1(P_N)^{n-1} < 0$$

  where $\mathcal{V}_*$ is the naturally induced parabolic structure on $\mathcal{V}$

Proof: the Bogomolov-Gieseker inequality

- The Bogomolov-Gieseker inequality for stable parabolic bundles (proved by T. Mochizuki in 2006, *Astérisque*) asserts that

$$p(N) := \text{par-ch}_2(\mathcal{E}_*) \cdot c_1(P_N)^{n-2} \leq 0$$

- $p(N)$ a polynomial of degree $n-2$ in N

- Calculation of $\text{par-ch}_2(\mathcal{E}_*)$ and cup products in $H^*(X, \mathbb{Z})$ show

$$p(N) = Q(\mathbf{a}) \cdot N^{n-2} + O(N^{n-3})$$

- Since $p(N) \leq 0$ for $N \gg 1$, we must have $Q(\mathbf{a}) \leq 0$ $\square$

Proof of the Stability Theorem

Baby case: $T\mathbb{CP}^n$ is stable

Proof: let $\mathcal{V} \subset T\mathbb{CP}^n$ be a distribution of index $\imath = c_1(\mathcal{V})$ and rank r
Let $P \subset \mathbb{CP}^n$ be a generic line so that:

$$\mathcal{V}|_P = \mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(d_r)$$

is a vector subbundle of $T\mathbb{CP}^n|_P = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ and P is transversal to $\mathcal{V}$, i.e., $TP \not\subset \mathcal{V}|_P$ then $d_i \leq 1$ for all i and

$$c_1(\mathcal{V}) = d_1 + \ldots + d_r \leq r \quad \square$$

Key estimate

Suppose that $(\mathcal{H}, \mathbf{a})$ is a weighted arrangement that is klt and CY. Then there is $\delta > 0$ such that, for any distribution $\mathcal{V} \subset T\mathbb{CP}^n$ with index $\imath = c_1(\mathcal{V}) \geq 0$, we have

$$\sum_{H \mid H \pitchfork \mathcal{V}} a_H \geq \imath + \delta \,,$$

where the sum is over all $H \in \mathcal{H}$ that are transverse to $\mathcal{V}$.

Example. Let $M \subset \mathbb{CP}^n$ be a linear subspace with $\dim M = r - 1$ for some $1 \leq r \leq n - 1$. The collection of all r-dimensional subspaces that contain M defines a distribution $\mathcal{V} \cong \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus r}$ of index $\imath = r$.
A hyperplane is tangent to $\mathcal{V}$ if and only if $H \supset M$.

Application

Theorem (dB-Panov, 2024)

Suppose that for all $L \in \mathcal{L}$ *we have*

$$m(L) < \operatorname{codim} L \cdot \frac{k}{n+1} \quad \big(\, i.e.\ \mathbf{1} \in C^\circ \,\big)$$

Then

$$\sum_{L \in \mathcal{L}^{n-2}} m_L \geq \left(1 - \frac{2}{n+1}\right) k^2 + k \quad \big(\, i.e.\ Q(\mathbf{1}) \leq 0 \,\big)$$

Equality holds if and only if every $H \in \mathcal{H}$ *intersects* $\mathcal{H} \setminus \{H\}$ *along*

$$\left(1 - \frac{2}{n+1}\right) k + 1 \quad \big(\, i.e.\ \mathbf{1} \in \ker Q \,\big)$$

codimension 2 *subspaces.*

Hirzebruch arrangements

A hyperplane arrangement $\mathcal{H}$ is Hirzebruch if:

- for every subspace $L \in \mathcal{L}$ we have

$$\frac{m_L}{\operatorname{codim} L} \leq \frac{k}{n+1}$$

- every $H \in \mathcal{H}$ intersects the others along

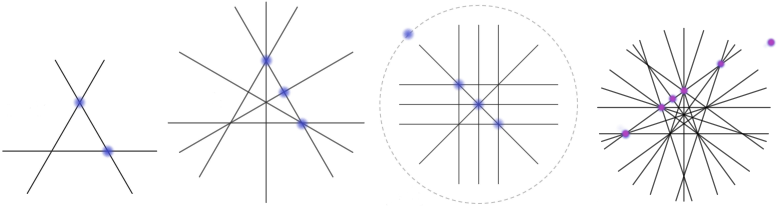
$$\left(1 - \frac{2}{n+1}\right) k + 1$$

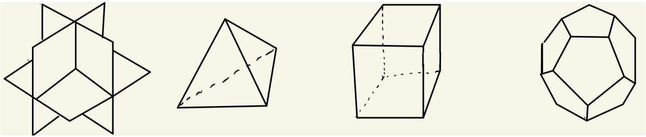
  codimension 2 subspaces.

Examples: (i) The $n+1$ coordinate hyperplanes in $\mathbb{CP}^n$.
(ii) If $n = 1$ and $|\mathcal{H}| \geq 2$ then $\mathcal{H}$ is Hirzebruch. If $n \geq 2$ then Hirzebruch arrangements are more rare
(iii) *Irreducible complex reflection arrangements*

Hirzebruch line arrangements/Geometrization

Hirzebruch's problem 1985. Consider an arrangement of k lines in $\mathbb{C}P^2$ such that each line intersects others in exactly $k/3+1$ points. Does such an arrangement consist of mirrors of a finite reflection group?





Panov (Geometry & Topology, 2018): there exist exactly four Hirzebruch line arrangements in $\mathbb{RP}^2$

THANK YOU!