Polyhedral Kähler cone metrics on $\mathbf{C}^n$

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Introduction

- **Long term goal:** produce (and classify) constant holomorphic sectional curvature metrics with cone singularities using (singular, i.e. logarithmic) flat connections.
- **Precedents:**
  - Smooth case. Hitchin gave a proof of the Uniformization Theorem (existence of hyperbolic Kähler metrics on compact Riemann surfaces) using the theory of Higgs bundles.
    Simpson extended the method to higher dimensions. In particular, he used Higgs bundles to prove the Miyaoka-Yau characterization of compact complex hyperbolic surfaces.
  - Panov used the parabolic Kobayashi-Hitchin correspondence to classify **polyhedral Kähler** (PK) metrics on $\mathbf{CP}^2$ with cone angles $< 2\pi$ along line arrangements.
- **Key technical point:** analyse connections of the form

$$\nabla = d - \sum_{H \in \mathcal{H}} A_H \frac{dh}{h}$$

2-dim models

Fix $\alpha > 0$. The models

$$\begin{cases} dr^2 + \alpha^2(\sin^2 r)d\theta^2 \\ dr^2 + \alpha^2 r^2 d\theta^2 \\ dr^2 + \alpha^2(\sinh^2 r)d\theta^2 \end{cases}$$

define constant curvature metrics with cone angle $2\pi\alpha$ at the origin.

Examples.

- Orbifold quotients $\alpha = 1/k$, pull-backs $\alpha = k$
- Doubles of (spherical/Euclidean/hyperbolic) polygons
- Polyhedral (spherical/Euclidean/hyperbolic) surfaces

Uniformization

S compact oriented surface with cone points $2\pi\alpha_1, \dots, 2\pi\alpha_N$

- Gauss-Bonnet: $a_i = 1 - \alpha_i$

$$\frac{1}{2\pi}\int_S K dA = \chi(S) - \sum_i a_i$$

- Anti-clockwise rotation by 90° gives a smooth complex structure

$$\{\text{const. curv., angles } 2\pi\alpha_i\} \xrightarrow{\Phi} \{\text{pairs } (X, \sum_i a_i \cdot x_i)\}$$

Quotient by isometries/scale and marked bihol.

Theorem (McOwen, Troyanov, Luo-Tian, ...)

Φ *defines a bijection if* $\chi(S,\alpha) \leq 0$ *or* $\chi(S,\alpha) > 0$ *where:*
$0 < \alpha_i < 1$ (so $X \cong \mathbf{CP}^1$) *and* $a_i < \sum_{j \neq i} a_j$ *with* $N \geq 3$

Example: spherical triangles

Let T be a spherical triangle with interior angles $\pi\alpha_i$ with $0 < \alpha_i < 1$

$$\text{area}(T) = \pi(\alpha_1 + \alpha_2 + \alpha_3 - 1)$$

T is the intersection of 3 spherical wedges W_i with

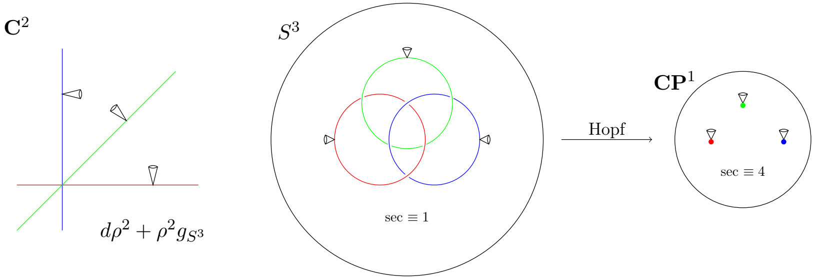
$$\text{area}(W_i) = 2\pi\alpha_i$$

The inequalities $\text{area}(T) > 0$ together with $\text{area}(T) < \text{area}(W_i)$ define a simplex inside the unit cube $Q = \{0 < \alpha_i < 1\}$ given by

$$\sum_i a_i < 2, \ \ a_i < a_j + a_k$$

Reflecting the simplex across the faces of Q and it translates gives all possible spherical triangles with arbitrary large (non-integer) angles

PK cone metrics on $\mathbf{C}^2$



- **Regular** $\iff$ spherical metrics on $\mathbf{CP}^1$ with cone points
  Restriction to any complex line through 0 is a 2-cone $C(2\pi\alpha_0)$ with

$$\alpha_0 = \frac{1}{2}\left(2 - \sum_i a_i\right) \qquad \text{(holonomy Hopf} = 2 \text{ area)}$$

- Quasi-regular: pull-back of regular by $(z, w) \mapsto (z^p, w^q)$
- Irregular: $C(2\pi\alpha_1) \times C(2\pi\alpha_2)$ with α_1/α_2 irrational

Non-negatively curved PK metrics on $\mathbf{CP}^2$

- $L \subset \mathbf{CP}^2$ complex lines with weights $0 < a_L = 1 - \alpha_L < 1$
- For every multiple point x of the arrangement with multiplicity $\mu_x \geq 3$ assume $\sum_{L|x \in L} a_L < 2$ and set $2a_x = \sum_{L|x \in L} a_L$

Theorem (Panov, 2009)

Suppose that $\sum a_L = 3$ *and for all* L

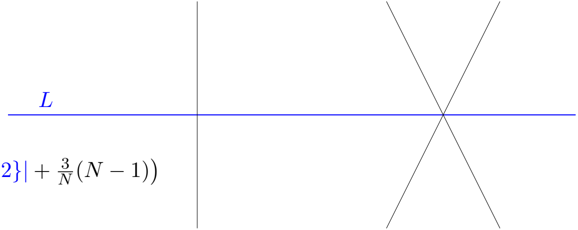
$$\sum_{L' | L' \pitchfork L} a_{L'} + \sum_{x \in L | \mu_x \geq 3} a_x = 2 \tag{GB}$$

Then there exists a (unique up to scale) PK metric on $\mathbf{CP}^2$ *with cone angles* $2\pi\alpha_L$ *along* L. *(These numerical conditions are also necessary.)*

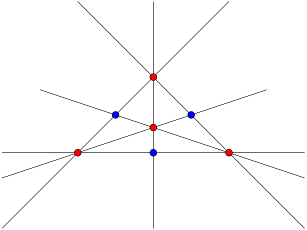
Rmk: Non-negatively curved polyhedral 4-manifold M with irreducible holonomy and $b_2 > 0$ is necessarily Kähler and $M \cong \mathbf{CP}^2$

Example: Hirzebruch line arrangements

$a_L = 3/N$ for all L



$$2 = \frac{1}{2}\left(\frac{3}{N}\,|\{x \in L | \mu_x \geq 2\}| + \frac{3}{N}(N-1)\right)$$

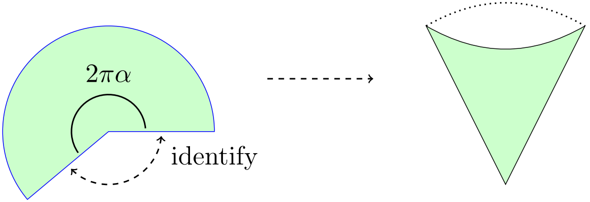


- Every line intersects the others at $(N/3)+1$ points
- Complex reflection groups
- Complete quadrilateral $(N = 6)$

Linear coordinates: the 2-cone

$C(2\pi\alpha)$ with metric $dr^2 + \alpha^2 r^2 d\theta^2$ has complex coordinate $z = r^{1/\alpha} e^{i\theta}$

$$\mathbf{C}_\alpha = \left(\mathbf{C}, |z|^{-2a}|dz|^2\right), \quad a = 1 - \alpha$$



- Levi-Civita connection $\nabla = d + \partial \log |s|^2$ (with $s = \partial/\partial z$) is

$$\nabla = d - a\frac{dz}{z}$$

- Euler vector field $e = (1/\alpha)\partial/\partial z$

Linear coordinates on PK cones: $\dim_{\mathbf{C}} = 2$

Regular PK cone $\iff$ spherical metric with cone points at $L_i \in \mathbf{CP}^1$
Linear complex coordinates (z, w) on $\mathbf{C}^2$ such that:

- $L_i = \{\ell_i(z, w) = 0\}$ complex lines through 0
- Euler vector field is $e = (1/\alpha_0)\,(\partial/\partial z + \partial/\partial w)$
- Levi-Civita connection in the $\partial/\partial z, \partial/\partial w$ trivialization is

$$\nabla = d - \sum_i A_i \frac{d\ell_i}{\ell_i}$$

where A_i are constant matrices with

$$\mathrm{tr}(A_i) = a_i, \;\; \ker A_i = L_i, \;\; \sum_i A_i = a_0 \cdot \mathrm{Id}$$

Standard connections

$\mathcal{H}$ finite collection of hyperplanes $H \subset \mathbf{C}^n$ going through the origin
A **standard connection** in the frame $\partial/\partial z_1, \ldots, \partial/\partial z_n$ has the form

$$\nabla = d - \sum_{\mathcal{H}} A_H \frac{dh}{h}$$

where A_H are non-zero constant matrices called *residues*

- ∇ is torsion free $\iff$ $\ker A_H = H$
- ∇ is flat $\iff$ $[A_L, A_H] = 0$ $\forall L \subset H$ (enough $\operatorname{codim} L = 2$)

$$\text{Notation: } A_L = \sum_{H | L \subset H} A_H, \;\; A_H(v) = dh(v) \cdot n_H$$

Obstruction: $H' \pitchfork H$ implies $[A_H, A_{H'}] = 0$ hence $\mathbf{C} \cdot n_H \subset H'$

Example: Lauricella connection ∇^a

- Braid arrangement $H_{ij} = \{z_i = z_j\} \subset \mathbf{C}^{n+1}$
- $a = (a_1, \ldots, a_{n+1}) \in \mathbf{C}^{n+1}$ parameters
- Jordan-Pochammer matrices
  $A_{ij}(\vec{\mathbf{v}}) = (v_i - v_j)(a_j \vec{\mathbf{e}}_i - a_i \vec{\mathbf{e}}_j)$
- Quotient by main diagonal $\mathbf{C} \cdot (1, \ldots, 1)$

$$\begin{array}{r} \\ \\ \\ \\ i \rightarrow \\ \\ \\ \\ j \rightarrow \\ \\ \\ \\ \end{array}\overset{\qquad\qquad\qquad i\downarrow \qquad\qquad\qquad\quad j\downarrow}{\begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & a_j & 0 & \cdots & 0 & -a_j & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -a_i & 0 & \cdots & 0 & a_i & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}}$$

$$\nabla^a = d - \sum_{i<j} A_{ij} \frac{d(z_i - z_j)}{z_i - z_j}$$

- $\ker A_{ij} = H_{ij}$ and $\mathrm{tr}(A_{ij}) = a_i + a_j$
- $[A_{ij}, A_{kl}] = 0$ if $\{i, j\} \cap \{k, l\} = \emptyset$ and $[A_{ij}, A_{ik} + A_{jk}]$ if $i < j < k$
- $(n = 2)$ 3 lines in $\mathbf{C}^2$ and $(n = 3)$ 6 planes in $\mathbf{C}^3$ (complete quad.)

Main result

∇ standard flat torsion free **unitary** connection i.e. $\mathrm{Hol}(\nabla) \subset U(n)$

- Non-integer: $a_H = \operatorname{tr} A_H \in \mathbf{R} \setminus \mathbf{Z}$
- Positivity: $a_L = (\operatorname{codim} L)^{-1} \sum_{L \subset H} a_H < 1$ (i.e. $\alpha_L = 1 - a_L > 0$)

Theorem (dB-Panov, 2021)

The metric completion is a polyhedral Kähler cone metric on $\mathbf{C}^n$ with cone angles $2\pi\alpha_H$ along H

- We allow $\alpha_H > 1$
- Quotient by $\mathbf{C}^*$ polyhedral Fubini-Study metrics
- The volume form of the PK cone is $\left(\prod |h|^{-2a_H}\right) dz \wedge d\bar{z}$
- Positivity $\iff \prod |h|^{-2a_H}$ is locally integrable

Application: the braid arrangement

Recall the (essential) braid arrangement is made of $\binom{n+1}{2}$ hyperplanes

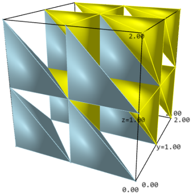
$$H_{ij} = \{z_i = z_j\} \subset \mathbf{C}^{n+1}/\mathbf{C} \cdot (1, \dots, 1)$$

For each $a = (a_1, \dots, a_{n+1})$ there is a **Lauricella connection** ∇^a

- If $a \in \mathbf{R}^{n+1}$ and $\{a_i\} \neq 0$ for all i then ∇^a admits a (unique up to scale) non-degenerate invariant Hermitian form
- The signature is $(p, n-p)$ with

$$p = \lfloor \sum_{i=1}^{n+1} \{a_i\} \rfloor \qquad \text{(Deligne-Mostow)}$$

- $n = 2 \to$ sphere with three cone points



Proof

1. Induction on n. For $n = 1$ we have $\nabla = d - adz/z$ and positivity $\alpha = 1 - a > 0$ is all we need
2. Enough to show *local product decomposition* away from the origin (characterization of polyhedral metrics by Lebedeva-Petrunin)
3. Close to $x \in L$ write $\nabla = \nabla^L + (\text{hol.})$ with ∇^L standard and splits according to the product decomposition of $\mathcal{H}_L = \{H \in \mathcal{H} | L \subset H\}$
4. Holomorphic frame $G : U \setminus \mathcal{H} \to GL(n, \mathbf{C})$ such that $G^* \nabla^L = \nabla$ (semi-simple representations have closed conjugate orbits)
5. G extends holomorphically over $\mathcal{H}$ (non-integer condition, normal form for logarithmic connections)
6. **Local dilation vector field** $e_x = e_0 - s$ where s is a parallel vector field $\nabla s = 0$ with $s(x) = e_0(x)$ (constructed using G)
7. Linearise e_x (Poincaré-Dulac)

THANK YOU!