Parabolic bundles and convex spherical metrics

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4 March, USTC

Background: Troyanov/Luo-Tian Theorem

Fix $n \geq 3$ points $x_i \in \mathbf{CP}^1$ together with real numbers $0 < \alpha_i < 1$ s.t.

$$\sum_{i=1}^{n}(1-\alpha_i) < 2 \qquad \text{(Gauss-Bonnet)}$$

$$(1-\alpha_i) < \sum_{j \neq i}(1-\alpha_j) \qquad \text{(Stability)}$$

Theorem (Troyanov, Luo-Tian)

There exists a unique conformal spherical metric g on $\mathbf{CP}^1$ with cone angles $2\pi\alpha_i$ at x_i

- Outside $\{x_i\}$ we can find local complex coordinate z s.t.

$$g = \frac{4}{(1+|z|^2)^2}|dz|^2 \tag{1}$$

- At x_i we can find centred complex coordinate z such that g is equal to the pull-back of (1) by $z \mapsto z^{\alpha_i}$

Introduction

- Joint work with Dmitri Panov: prove the Troyanov/Luo-Tian Theorem using **parabolic bundles**
- **Precedents**: Lingguang Li, Jijian Song, and Bin Xu: *Irreducible cone spherical metrics and stable extensions of two line bundles* Semin Kim and Graeme Wilkin: *Analytic convergence of harmonic metrics for parabolic Higgs bundles*
- **Motivation**: Extend the method to higher complex dimensions. Produce conical Fubini-Study metrics and (introducing a Higgs field) conical complex hyperbolic metrics. (Cone angles $< 2\pi$) E.g. Carlos Simpson: *Constructing variations of Hodge structures using Yang-Mills theory and application to uniformization* (smooth case, complex hyperbolic surfaces)

Parabolic bundles

- X compact Riemann surface with marked points $\{x_i\}_{i=1}^n \subset X$
- E holomorphic vector bundle over X

Definition (Parabolic structure E_* on E)

$\{E_a^i\}$ locally free sheaves ($1 \leq i \leq n$ and $a \in \mathbf{R}$) with $E_1^i = E$ s.t:

(i) Increasing filtration: $E_a^i \subset E_{a'}^i$ for $a < a'$
(ii) Semi-continuity: $E_{a+\epsilon}^i = E_a^i$ for $0 < \epsilon \ll 1$
(iii) Periodicity: $E_{a-1}^i = E_a^i \otimes \mathcal{O}(-x_i)$

- Enough to know E_a^i for $0 < a \leq 1$
- $E_a^i = E$ on $X \setminus \{x_i\}$
- The quotients $E_a^i / E_{<a}^i$ are skycraper sheaves supported at x_i
- $E_* \iff$ *flags* at the fibres E_{x_i} together with weights

Parabolic degree and stability

Recall $\deg(E) = \int_X c_1(E)$

Definition (Parabolic degree)

$$\text{par-deg}(E_*) = \deg(E) - \sum_i \sum_a a \cdot \operatorname{rk}(E^i_a / E^i_{<a})$$

Definition (Parabolic stability)

E_* is stable if for every non-zero locally free sheaf $V \subsetneq E$ we have

$$\frac{\text{par-deg}(V_*)}{\operatorname{rk} V} < \frac{\text{par-deg}(E_*)}{\operatorname{rk} E}$$

where V_* is the induced parabolic structure given by $V^i_a = V \cap E^i_a$

The parabolic structure

- Let $\{x_i\}_{i=1}^n \subset \mathbf{CP}^1$ together with $0 < \alpha_i < 1$ as in Troy./Luo-Tian
- The vector bundle: $E = \mathcal{O}(1) \oplus \mathcal{O}(n-1)$ over $X = \mathbf{CP}^1$
- Weights: $0 < a_{i1} < a_{i2} < 1$ given by

$$a_{i1} = \frac{1-\alpha_i}{2}, \quad a_{i2} = \frac{1+\alpha_i}{2}$$

- **Flags**: $\{0\} \subsetneq F_i \subsetneq E_{x_i}$ in the fibres E_{x_i} given by

$$F_i = \mathcal{O}(1)_{x_i} \text{ for } i = 1, \dots, n-1$$
$$F_n = \mathbf{C} \cdot v_n \text{ with } v_n \notin \mathcal{O}(1)_{x_n} \text{ and } v_n \notin \mathcal{O}(n-1)_{x_n}$$

- The parabolic structure E_* is given by

$$E_a^i = \begin{cases} \text{sections of E that vanish at } x_i \text{ for } 0 < a \leq a_{i1} \\ \text{sections of E that are tangent to } F_i \text{ for } a_{i1} \leq a < a_{i2} \\ \text{sections of E for } a_{i2} \leq a \leq 1 \end{cases}$$

Stability Proposition

par-deg $E_* = 0$ *Proof*: par-deg $E_* = \deg E - \sum_i (a_{i1} + a_{i2})$ while $\deg E = n$ and $a_{i1} + a_{i2} = 1$ for each i □

Stability Proposition

E_* is stable, i.e., $\forall$ line sub-bundle $L \subset E \implies$ par-deg $L_* < 0$

$$\text{par-deg}\, L_* = \deg L - \sum_{F_i \subset L_{x_i}} a_{i1} - \sum_{F_i \not\subset L_{x_i}} a_{i2} \qquad (*)$$

Since par-deg $L_* < \deg L$ we can assume $\deg L > 0$

Auxiliary Lemma: line sub-bundes $L \subset E$ of positive degree

- If $\deg L > 1$ then $L = \mathcal{O}(n-1)$
- If $\deg L = 1$ then there is at least one $1 \leq i \leq n$ s.t. $F_i \not\subset L_{x_i}$

Proof of Stability Proposition

- If $L = \mathcal{O}(n-1)$ then $F_i \not\subset L_{x_i}$ for all i and

$$\begin{aligned} \text{par-deg}\, L_* &= n - 1 - \frac{1}{2}\sum_i (1+\alpha_i) \\ &= -1 + \frac{1}{2}\sum_i (1-\alpha_i) < 0 \qquad \text{(by Gauss-Bonnet)} \end{aligned}$$

- If $\deg L = 1$ let $1 \leq i \leq n$ s.t. $F_i \not\subset L_{x_i}$ then

$$\begin{aligned} \text{par-deg}\, L_* &\leq 1 - \frac{\alpha_i + 1}{2} - \frac{1}{2}\sum_{j \neq i} (1-\alpha_j) \\ &= \frac{1-\alpha_i}{2} - \frac{1}{2}\sum_{j \neq i}(1-\alpha_j) < 0 \qquad \text{(by Stability)} \end{aligned}$$

□

Proof of Auxiliary Lemma

Compose $L \subset E$ with projections to get

$$\sigma_1 \in H^0(L^* \otimes \mathcal{O}(1)), \quad \sigma_2 \in H^0(L^* \otimes \mathcal{O}(n-1))$$

- If $\deg L > 1$ then $\deg(L^* \otimes \mathcal{O}(1)) < 0$ so $\sigma_1 = 0$ and $L = \mathcal{O}(n-1)$
- If $\deg L = 1$ then σ_1 is an isomorphism and

$$s = \sigma_2 \circ \sigma_1^{-1} \in H^0(\mathcal{O}(n-2))$$

If $F_i = \mathcal{O}(1)_{x_i} \subset L_{x_i}$ for $i = 1, \ldots, n-1$ then $s(x_i) = 0$ for $i = 1, \ldots, n-1$. Hence $s = 0$, so $L = \mathcal{O}(1)$ and $F_n \not\subset L_{x_n}$ □

$$\mathrm{Aut}\, E = \left\{ \begin{pmatrix} \lambda_1 & 0 \\ P & \lambda_2 \end{pmatrix} \text{ with } \lambda_1, \lambda_2 \in \mathbf{C}^* \text{ and } P \in H^0(\mathcal{O}(n-2)) \right\}$$

$\mathbf{F} = \{F_1, \ldots, F_n\}$ is uniquely characterized (up to $\mathrm{Aut}(E)/C^*$) by: $F_i \cap \mathcal{O}(n-1) = \{0\}$ $\forall i$ and $\nexists L \subset E$ with $\deg L = 1$ and $\mathbf{F} \subset L$

Logarithmic connections

- X compact Riemann surface $D = x_1 + \ldots + x_n$ divisor
- $\Omega^1(\log D)$ meromorphic 1-forms with simple poles at x_i
- E holomorphic vector bundle over X

Definition (Logarithmic connection)

$\mathbf{C}$-linear $\nabla : E \to E \otimes \Omega^1(\log D)$ with $\nabla(fs) = df \cdot s + f\nabla s$

Complex coordinate t centred at x_i and w.r.t. holomorphic frame

$$\nabla = d - A(t)\frac{dt}{t}$$

Definition (Residue)

$$\mathrm{Res}_{x_i}(\nabla) = A(0) \in \mathrm{End}(E_{x_i})$$

Mehta-Seshadri Theorem

Definition (Log. connect. compatible with parabolic structure)

∇ compatible with E_* if

(i) $\mathrm{Res}_{x_i}(\nabla)$ preserve the subspaces $E^i_a/E^i_0 \subset E_{x_i}$ for all $0 < a \leq 1$

(ii) $\mathrm{Res}_{x_i}(\nabla)$ acts on $E^i_a/E^i_{<a}$ as scalar multiplication by a

(iii) ∇ restricts to logarithmic connection on E^i_a for all $0 < a \leq 1$

Parabolic Kobayashi-Hitchin correspondence in complex dimension 1:

Theorem (Mehta-Seshadri)

If par-deg $E_* = 0$ *and* E_* *is stable then it admits a unique* ***unitary*** *logarithmic connection compatible with* E_*

Rmk: Logarithmic $\implies$ Flat (because $\dim_{\mathbf{C}} X = 1$)

The connection

Recall:

- $x_i \in \mathbf{CP}^1$, $0 < \alpha_i < 1$
- $E = \mathcal{O}(1) \oplus \mathcal{O}(n-1)$ with parabolic structure E_* given by

(i) Flags $\mathbf{F} = \{F_1, \ldots, F_n\}$ with $F_i = \mathcal{O}(1)_{x_i}$ for $1 \le i \le n-1$ and $F_n \not\subset \mathcal{O}(1)_{x_n} \cup \mathcal{O}(n-1)_{x_n}$

(ii) Weights $0 < a_{i1} < a_{i2} < 1$ with $a_{i1} = \frac{1-\alpha_i}{2}$ and $a_{i2} = \frac{1+\alpha_i}{2}$

Apply Mehta-Seshadri to E_* to obtain:

- ∇ logarithmic connection on E
- $F_i = a_{i1}$-eigenspace of $\mathrm{Res}_{x_i}(\nabla)$
- Holomorphic trivialization close to x_i

$$\nabla = d - \begin{pmatrix} a_{i1} & 0 \\ 0 & a_{i2} \end{pmatrix} \frac{dt}{t} \qquad (**)$$

Note: $a_{i2} - a_{i1} = \alpha_i \notin \mathbf{Z} \implies$ non-resonant Fuchsian singularity

The foliation

- $\mathbf{P}(E)$ is the Hirzebruch surface

$$\Sigma_{n-2} = \mathbf{P}\left(\mathcal{O} \oplus \mathcal{O}(n-2)\right)$$

- $U = \mathbf{CP}^1 \setminus \{x_1, \dots, x_n\}$
- $\nabla \implies$ horizontal distribution on $E|_U \implies$ foliation $\mathcal{F}$ on $\mathbf{P}(E)|_U$
- Leaves of $\mathcal{F}$ are locally given by projecting flat sections

Extension Lemma

$\mathcal{F}$ extends to a singular foliation on Σ_{n-2} tangent to $\mathbf{P}(E)_{x_i}$ with two singularities at $v_{i1} = F_i$ and $v_{i2} = a_{i2}$-eigenspace of $\mathrm{Res}_{x_i}(\nabla)$

Proof: Use (**) flat sections

$$t \mapsto \begin{pmatrix} y_1 = c_1 t^{a_{i1}} \\ y_2 = c_2 t^{a_{i2}} \end{pmatrix} \text{ with } c_1, c_2 \in \mathbf{C}$$

$y = y_1/y_2 = ct^{-\alpha_i}$ are flat sections of $d + (\alpha_i/t)dt$. Similarly y_2/y_1 □

The section

$\mathcal{O}(n-1) \subset E$ defines a section σ of $\mathbf{P}(E)$ with $\sigma^2 = 2 - n$

- σ is everywhere transversal to $\mathcal{F}$

Proof: The total number of tangencies of $\mathcal{F}$ with a curve is

$$\mathrm{Tan}(\mathcal{F}, C) = C^2 - C \cdot T_{\mathcal{F}}$$

Lifting a holomorphic vector field shows that $T_{\mathcal{F}} = (2-n) \cdot \mathfrak{f}$ where $\mathfrak{f} =$ class of a fibre. Since $\sigma \cdot \mathfrak{f} = 1$ (because s is a section) we obtain

$$\mathrm{Tan}(\mathcal{F}, \sigma) = \sigma^2 - \sigma \cdot T_{\mathcal{F}} = 0 \quad \square$$

The spherical metric

Use σ to pull-back the Fubini-Study metrics on the fibres:

- e_1, e_2 parallel unitary frame on $V \subset \mathbf{CP}^1 \setminus \{x_1, \dots, x_n\}$
- $\sigma \iff f : V \to \mathbf{CP}^1$ holomorphic
- $df(x) \neq 0 \; \forall x \in V$ because σ is transverse to $\mathcal{F}$
- $g = f^* g_{\mathbf{CP}^1}$ frame independent because $g_{\mathbf{CP}^1}$ is $U(2)$-invariant

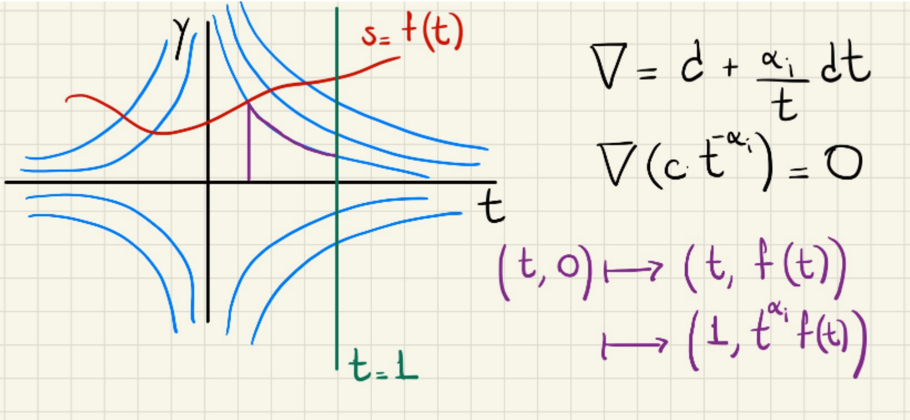
Cone angles

The conformal spherical metric g on $\mathbf{CP}^1 \setminus \{x_1, \dots, x_n\}$ extends over x_i with cone angle $2\pi\alpha_i$

Proof: $\mathcal{F}$ = flat sections of $d + (\alpha_i/t)dt$ and $\sigma = f(t)$ with $f(0) \neq 0$

$$(t, 0) \mapsto (t, f(t)) \xrightarrow{\text{leaves of } \mathcal{F}} (1, t^{\alpha_i} f(t))$$

Take $z = tf(t)^{1/\alpha_i}$ so g is the pull-back of $g_{\mathbf{CP}^1}$ under $z \mapsto z^{\alpha_i}$ □



THANK YOU!