ALE metrics with cone singularities along a divisor

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July, 2012

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1 Introduction

This transfer is motivated by two main questions.

$Question 1$: Let C be a smooth curve in $\mathbb{C}^2$. Does there exist a Kahler Ricci flat metric with a cone singularity along C which is Asymptotically Locally Euclidean (ALE)?

$Question 2$: In the case that such a metric exist, can we compute the energy?

Let X be a complex manifold of complex dimension n, $D \subset X$ a smooth divisor and $\beta \in (0, 1)$. We say that g is a Kahler metric with a cone singularity along D of angle β if g is a smooth Kahler metric on $X \setminus D$ and for every coordinate chart $(z_1, \ldots, z_n)$ in which $D = \{z_1 = 0\}$ we can find $C > 0$ such that $C^{-1} g_\beta \leq g \leq C g_\beta$. Where

$$g_\beta = |z_1|^{2\beta-2}|dz_1|^2 + \sum_{j=2}^{n} |dz_j|^2$$

Let us be more precise with the meaning of ALE in this context.

Write $C = \{p = 0\}$ where f is a polynomial of degree d in two variables. Denote by p_d the homogeneous part of degree d of p. We restrict to the generic case in which the zero set of p_d is composed by

d different lines, i.e, C has d different asymptotic lines. This d lines $l_1, \ldots, l_d$ correspond to d points in $\mathbb{CP}^1$, denoted by $p_1, \ldots, p_d$.

Suppose that there is a metric $\overline{g}$ on $\mathbb{CP}^1$ with constant positive curvature and cone singularities of angle β at $p_1, \ldots, p_d$.

Consider the Hopf map $h : S^3 \to \mathbb{CP}^1$.

If α is a 1-form on S^3 which does not vanish on the fibers of h, we can define a metric on S^3 by $\tilde{g} = h^*\overline{g} + \alpha^2$. By choosing the right α we can achieve that $\tilde{g}$ is locally isometric to $S^3(1)$ away of the fibers over the points $p_1, \ldots, p_d$.

Next we do the cone over $\tilde{g}$, i.e, $g = d\rho^2 + \rho^2\tilde{g}$. We can also define an orthogonal complex structure I on $\mathbb{R}_{>0} \times S^3$ by lifting horizontally the complex structure on $\mathbb{CP}^1$ and sending the Reeb vector field to $\partial/\partial\rho$. It is a fact that $(\mathbb{R}_{>0} \times S^3, g, I)$ is a Kahler manifold. Moreover, we can find holomorphic functions z, w which identify $(\mathbb{R}_{>0} \times S^3, I)$ with $\mathbb{C}^2$. The metric is flat and has cone angle β along $l_1, \ldots, l_d$. If $\omega = g(I., .)$ is the corresponding Kahler form, then

$$\omega^2 = |l_1|^{2\beta-2} \ldots |l_d|^{2\beta-2} \Omega \wedge \overline{\Omega}$$

where $\Omega = dz \wedge dw$.

We refer to these metrics as background metrics and use the notation g_0, ω_0 for them.

Let us mention that there is a restriction for the existence of $\overline{g}$ coming from the Gauss-Bonnet formula which gives

$$\beta > \frac{d-2}{d} \tag{1.1}$$

Now we can formulate $Question 1$ more precisely.

Does there exist a Kahler metric g on $\mathbb{C}^2$ with a cone singularity along $C = \{f = 0\}$ such that

1) g is Ricci flat. I.e

$$\omega^2 = |p|^{2\beta-2} \Omega \wedge \overline{\Omega}$$

2) There exist a compact set $K \subset \mathbb{C}^2$ and a diffeomorphism $\phi : \mathbb{C}^2 \setminus K \to \mathbb{C}^2 \setminus B_R$ such that for some $\gamma < 0$

$$|\phi_* g - g_0| = O(\rho^\gamma) \tag{1.2}$$

$$|\phi_* I - I| = O(\rho^\gamma) \tag{1.3}$$

The norms are taken with respect to g_0. I denotes the complex structure of $\mathbb{C}^2$.

1.2 implies that the image under ϕ of C is the intersection of the d lines with the complement of B_R. If ϕ is holomorphic, by Hartago's theorem it extends to a biholomorphism. But then the image of C would have to be a smooth curve, so instead of requiring $\phi_* I = I$ we relax to 1.3 .

Let us now explain the organization of the transfer.

We start with a detailed exposition of the Gibbons Hawking ansatz in section 2. This ansatz produces an hyperkahler 4 manifold M_0 from a positive harmonic function f on a domain U in $\mathbb{R}^3$. More precisely M_0 is the total space of an S^1 bundle over U with a connection $i\alpha$ such that

$$d\alpha = - \star df$$

and the hyperkahler metric is given by

$$g = f g_{\mathbb{R}^3} + f^{-1}\alpha^2$$

.

The sphere of complex structures of the hyperkahler structure on M_0 is naturally identified with the unit sphere in $\mathbb{R}^3$.

We show in subsections 2.2 and 2.3 that when

$$f(x) = c + \sum_{i=1}^{k} \frac{1}{2|x - p_i|}, \qquad c \geq 0$$

and $U = \mathbb{R}^3 \setminus \{p_1, \dots, p_k\}$, the space M_0 together with the hyperkahler structure can be extended to $M = M_0 \sqcup \{\tilde{p_1}, \dots, \tilde{p_k}\}$ to give a complete hyperkahler manifold. Moreover, when $c = 0$, (M, g) is asymptotic to $\mathbb{C}^2/\mathbb{Z}_k$ where $e^{2\pi i/k}(z, w) = (e^{2\pi i/k}z, e^{-2\pi i/k}w)$.

In subsections 2.4 and 2.5 we study the complex structures coming from the ansatz. We show that if the complex structure I on (M, g) comes from a unit vector v in $\mathbb{R}^3$ which is not parallel to any of the segments connecting the points p_i then (M, I) can be identified with $\{zw = P(t)\} \subset \mathbb{C}^3$. Where $P(t) = (t - a_1)\dots(t - a_k)$ and a_i is the projection of p_i to $v^\perp$.

Finally in subsection 2.6 we compute the curvature tensor of (M, g) and derive the formula

$$|Rm|^2 = \frac{1}{4} f^{-1} \triangle \triangle f^{-1} \tag{1.4}$$

Integrating this we get that when $c = 0$

$$E = \int |Rm|^2 = 8\pi^2 (k - \frac{1}{k})$$

and when $c > 0$

$$E = 8\pi^2 k$$

In section 3 we describe the background metrics that we mentioned at the beginning. In subsection 3.1 we start by recalling that the Hopf map is a Riemannian submersion between the unit three sphere and the two sphere of radius 1/2. Using this description we write the euclidean metric on $\mathbb{C}^2$ as a cone over $S^3(1)$

Cone metrics appear for the first time in subsection 3.2. We state (without proof) the fundamental Theorem 1, due to Troyanov [4], on which our construction is based. The theorem says that if $\beta \in (0, 1)$ and $d \in \mathbb{N}$ satisfy 1.1 then there exist a metric with constant positive curvature on S^2 and cone angle β along any given d points.

Subsection 3.3 is a rewriting of 3.1 in a different order and in the context of cone singularities. We get a recipe for constructing the background metrics which goes as follows: Let $l_j = \{z = a_j w\}$ for $j = 1, \dots, d-1$ and $l_d = \{w = 0\}$ be d lines in $\mathbb{C}^2$. Assume that 1.1 holds, i.e

$$c = 2 + d\beta - d > 0$$

Write $\xi = z/w$. A metric on $\mathbb{CP}^1$ with constant curvature 4 and con angle β at $a_1, \dots, a_{d-1}, \infty$ is given by $\overline{g} = e^{2\phi}|d\xi|^2$. Where

$$-e^{-2\phi} \triangle \phi = 4$$

and the condition on the cone angles means that

$$u = \phi - (\beta - 1) \sum_{j=1}^{d-1} \log |\xi - a_j|$$

is continuous on $\mathbb{C}$ and $\phi + (1 + \beta) \log |\xi|$ is continuous at ∞.

Define a real function on $\mathbb{C}^2$ by

$$\rho^2 = |w|^c e^{-u} \tag{1.5}$$

Then

$$\omega = \frac{i}{2} \partial \overline{\partial} \rho^2$$

is a flat Kahler metric on $\mathbb{C}^2$ with angle β along the lines l_j that we started with.

We give examples of metrics solving $Question 1$ in section **??**. We start by examples coming from quotients of smooth metrics, thus giving examples with $\beta = 1/k$ and $k \in \mathbb{N}$, k ≥ 2. It follows from 1.1 that when the degree d of the curve C is greater or equal than 4 then $\beta > 1/2$ and therefore we do not expect examples coming from quotients of smooth metrics. When $d = 2$ there is not restriction on β and when $d = 3$ then $\beta \in (1/3, 1)$.

When the degree of C is 2 , after a linear change of coordinates, we can assume that $C = \{zw = 1\}$. We construct ALE Ricci flat Kahler metrics on $\mathbb{C}^2$ with angle $\beta = 1/k$ along C in subsection 4.1. We apply the Gibbons-Hawking ansatz to

$$f = \sum_{j=1}^{k} \frac{1}{2|x - p_j|}$$

with the points $p_j = (e^{2\pi ij/k}, 0) \in \mathbb{C} \times \mathbb{R} = \mathbb{R}^3$. We obtain a smooth hyperkahler manifold M asymptotic to $\mathbb{C}^2/\mathbb{Z}_k$. The complex structure I on M given by the $\mathbb{R}$ axis is that of $(M, I) \simeq \{zw = t^k + 1\} \subset \mathbb{C}^3$. You can lift the map that fixes the $\mathbb{R}$ axis and rotates $\mathbb{C}$ by an angle of $e^{2\pi i/k}$ as a map F from M to M, such that F preserves the connection and fixes the S^1 fibers over $0 \times \mathbb{R}$.

In the above complex description of (M, I) F is identified with $(z, w, t) \to (z, w, e^{2\pi i/k}t)$. Moreover the action of F on M is asymptotic to the action of $[(z, w)] \to [(e^{2\pi i/k}z, w)]$ on $\mathbb{C}^2/\mathbb{Z}_k$. It follows from these two facts that the quotient $M/ < F >$ is $\mathbb{C}^2$, the metric has cone angle $\beta = 1/k$ along C and has the asymptotic behavior that we want.

In subsection 4.2 we consider the case of curves of degree 3 and $\beta = 1/2$. We consider the group $D_4 \subset SU(2)$ generated by $(z, w) \to (iz, -iw)$ and $(z, w) \to (-w, z)$. The quotient of $\mathbb{C}^2/D_4$ by $[(z, w)] \to [(w, z)]$ is $\mathbb{C}^2$ with a metric of angle $\beta = 1/2$ along three lines. We show that there exist an hyperkahler metric on the surface $M = \{w^3 + wz^2 = 1 + t^2\}$ asymptotic to $\mathbb{C}^2/D_4$ such that $F(z, w, t) = (z, w, -t)$ preserves the metric and is asymptotic to $[(z, w)] \to [(w, z)]$. As above it follows that the quotient $M/ < F >$ is $\mathbb{C}^2$, the metric has cone angle $\beta = 1/2$ along $C = \{w^3 + wz^2 = 1\}$ and has the asymptotic behavior that we want.

In subsection 4.3 we come back to $C = \{zw = 1\}$ and, following Don [1] we construct metrics for all $\beta \in (0, 1)$ by applying the Gibbons-Hawking ansatz to the Green's function for the laplacian on the product of a cone of angle $2\pi\beta$ with $\mathbb{R}$. When $\beta = 1/k$ these are the metrics in 4.1.

In subsection 4.4 we use formula 1.4 together with some facts about the Green's function to compute the energy of the metrics constructed. The final answer being

$$E = 8\pi^2(1 - \beta^2)$$

Finally in section 5 we start a PDE way of answering $Question1$ which goes roughly as follows: given the curve $C = \{p = 0\}$ and $(d-2)/d < \beta < 1$ you construct a Kahler metric ω_0 on $\mathbb{C}^2$ and angle β along C with the desired asymptotic behavior. To do this you can use a diffeomorphism on the complement of a ball which maps C to the d asymptotic lines, you pullback the potential of the background metric. In the ball you have the potential $|z|^2 + |w|^2 + \epsilon|p|^{2\beta}$ with ϵ small. Then you glue these potentials to obtain ω_0.

$$\omega_0^2 = |p|^{2\beta-2}e^{-f}\Omega \wedge \overline{\Omega}$$

for some function f, which means that $\mathrm{Ric}(\omega_0) = i\partial\overline{\partial}f$. Then we modify ω_0 to $\omega = \omega_0 + i\partial\overline{\partial}\phi$ such that

$$(\omega_0 + i\partial\overline{\partial}\phi)^2 = e^f\omega_0^2 \tag{1.6}$$

if we achieve to find a solution to this equation with a function ϕ which preserves the asymptotic behavior of ω_0 and the cone singularities along C then ω is the solution to $Problem1$.

In subsection 5.1 we review some work of Joyce [2] on the equation 1.6 in the case that w_0 is smooth. In subsection 5.2 we put weights in a very standard way to the Schauder estimates in Don [1] as a first attempt to have a linear theory to solve 1.6.

2 Gibbons Hawking Construction

2.1 The euclidean metric on $\mathbb{R}^4$

On $\mathbb{R}^4$ with standard coordinates x_1, x_2, x_3, x_4 we have the euclidean metric

$$g = \sum_{i=1}^{4} dx_i^2.$$

We also have three orthogonal complex structures I, J, K satisfying the quaternionic relations $I^2 = J^2 = K^2 = IJK = -1$. These are given by

$$I\frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_2}, \qquad I\frac{\partial}{\partial x_3} = \frac{\partial}{\partial x_4}$$
$$J\frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_3}, \qquad J\frac{\partial}{\partial x_4} = \frac{\partial}{\partial x_2}$$
$$K\frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_4}, \qquad K\frac{\partial}{\partial x_2} = \frac{\partial}{\partial x_3}$$

coupled with g we obtain three symplectic forms

$$\omega_1 = g(I.,.) = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$$
$$\omega_2 = g(J.,.) = dx_1 \wedge dx_3 - dx_2 \wedge dx_4$$
$$\omega_3 = g(K.,.) = dx_1 \wedge dx_4 + dx_2 \wedge dx_3$$

These give to $\mathbb{R}^4$ the structure of a hyperkahler manifold. Note that $\omega_2 + i\omega_3 = dz \wedge dw$ where $z = x_1 + ix_2$ and $w = x_3 + ix_4$ are global complex coordinates for I. In the following we use these z and w to identify $\mathbb{R}^4$ with $\mathbb{C}^2$

The action of S^1

$$e^{it}(z, w) = (e^{it}z, e^{-it}w)$$

preserves g, I, J, K

Writing $z = r_1e^{i\theta_1}$ and $w = r_2e^{i\theta_2}$. The derivative of the action is

$$Y = \frac{\partial}{\partial \theta_1} - \frac{\partial}{\partial \theta_2}$$

and the (1,0) component is

$$Y^{(1,0)} = \frac{1}{2}(Y - iIY) = iz\frac{\partial}{\partial z} - iw\frac{\partial}{\partial w}$$

.

We look for the moments maps for this action $d\mu_i = \imath_Y \omega_i$.

Since $\omega_1 = r_1 dr_1 \wedge d\theta_1 + r_2 dr_2 \wedge d\theta_2$, we have $\imath_Y \omega_1 = -r_1 dr_1 + r_2 dr_2$ and we can take

$$\mu_1 = \frac{1}{2}(|w|^2 - |z|^2)$$

To compute the other two we use

$$d(\mu_2 + i\mu_3) = \imath_Y(\omega_2 + i\omega_3) = \imath_{Y^{(1,0)}} dz \wedge dw = izdw + iwdz = id(zw)$$

and we have

$$\mu_2 + i\mu_3 = izw$$

Writing $p = (\mu_1, \mu_3, \mu_3)$ we obtain a map from $\mathbb{R}^4$ to $\mathbb{R}^3$ whose fibers are the orbits of the S^1 action. This map restricts to an S^1 bundle, with the above action, from $\mathbb{R}^4 \setminus \{0\}$ to $\mathbb{R}^3 \setminus \{0\}$.

Now we look for a 1-form α such that $\alpha(Y) = 1$ and $\alpha = 0$ on the orthogonal complement of Y (i.e. $i\alpha$ is a connection for the distribution given by $Y^\perp = span\{IY, JY, KY\}$).

$\{d\mu_i, \alpha\}$ are pointwise linearly independent on $\mathbb{R}^4 \setminus \{0\}$, with dual frame given by $\{Y, |Y|^{-2}IY, |Y|^{-2}JY, |Y|^{-2}KY\}$. g and the symplectic forms ω_i are given in this coframe by

$$g = |Y|^{-2} \sum_{i=1}^{3} d\mu_i^2 + |Y|^2 \alpha \otimes \alpha$$

$$\omega_i = \alpha \wedge d\mu_i + |Y|^{-2} d\mu_j \wedge d\mu_k$$

where i, j, k run over cyclic permutations of $1, 2, 3$.

Also note that

$$\mu_1^2 + \mu_2^2 + \mu_3^2 = \left(\frac{|w|^2 - |z|^2}{2} \right)^2 + |z|^2 |w|^2 = \frac{(|z|^2 + |w|^2)^2}{4}$$

so if $|x|$ is the distance to 0 in $\mathbb{R}^3$ we have

$$(2|x|) \circ p = |z|^2 + |w|^2 = |Y|^2$$

.

Writing

$$f = \frac{1}{2|x|}$$

the equations for g and ω_i read

$$g = (f \circ p) \sum_{i=1}^{3} d\mu_i^2 + (f^{-1} \circ p) \alpha \otimes \alpha \tag{2.1}$$

$$\omega_i = \alpha \wedge d\mu_i + (f \circ p) d\mu_j \wedge d\mu_k \tag{2.2}$$

The equations $d\omega_i = 0$ for $i = 1, 2, 3$ are then equivalent to

$$d\alpha = -p^*(\star df) \tag{2.3}$$

which is known as the Bogomolony equation.

Finally let us compute the first Chern class of p seen as an S^1 bundle over $\mathbb{R}^3 \setminus \{0\}$. For this identify $H^2(\mathbb{R}^3 \setminus \{0\}, \mathbb{R})$ with $\mathbb{R}$ by integrating a 2 form over a sphere around 0, where we orient the sphere as the boundary of the ball that bounds. Then $H^2(\mathbb{R}^3 \setminus \{0\}, \mathbb{Z})$ gets identified with the forms integrating an integer number.

We know that $i\alpha$ is a connection on p. $F = d\alpha$ is the curvature and $c_1(p) = [\frac{i}{2\pi} F]$. From 2.3

$$c_1(p) = \frac{1}{2\pi} [\star df]$$

Computing

$$\frac{\partial f}{\partial x_i} = \frac{x_i}{2|x|^3}$$

so

$$2\star df|_{S^2(1)} = -x_1 dx_2 \wedge dx_3 - x_2 dx_3 \wedge dx_1 - x_3 dx_1 \wedge dx_2 = -\imath_{x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}} dx_1 \wedge dx_2 \wedge dx_3 = -dV_{S^2(1)}$$

and

$$c_1(p) = -1$$

On the other hand, we can get an explicit expresion for α

$$\alpha(X) = |Y|^{-2} g(X, Y)$$

and

$$\alpha^{(0,1)} = \alpha^{(0,1)}\left(\frac{\partial}{\partial\overline{z}}\right)d\overline{z} + \alpha^{(0,1)}\left(\frac{\partial}{\partial\overline{w}}\right)d\overline{w}$$

$$\alpha^{(0,1)}\left(\frac{\partial}{\partial\overline{z}}\right) = \alpha\left(\frac{\partial}{\partial\overline{z}}\right) = |Y|^{-2}g\left(Y, \frac{\partial}{\partial\overline{z}}\right) = i|Y|^{-2}g\left(z\frac{\partial}{\partial z} - w\frac{\partial}{\partial w}, \frac{\partial}{\partial\overline{z}}\right) = \frac{iz}{2(|z|^2+|w|^2)}$$

because $|Y|^2 = |z|^2 + |w|^2$ and $g(\frac{\partial}{\partial z}, \frac{\partial}{\partial\overline{z}}) = \frac{1}{2}$. So

$$\alpha^{(0,1)} = \frac{i}{2(|z|^2+|w|^2)}(zd\overline{z} - wd\overline{w})$$

Finally

$$\alpha = \alpha^{(0,1)} + \overline{\alpha^{(0,1)}} = 2Re\alpha^{(0,1)}$$

and using that for a 1 form γ , $Re(i\gamma) = Im(\overline{\gamma})$. We get

$$\alpha = Im\frac{\overline{z}dz - \overline{w}dw}{|z|^2+|w|^2} \tag{2.4}$$

With this formula for α you can, by long but straightforward computation, check 3, 2.2 and 2.3.

2.2 The ansatz

In the following we identify $H^2(\mathbb{R}^3 \setminus \{p_1, \dots, p_k\}, \mathbb{R})$ with $\mathbb{R}^k$ via integration on spheres centered at the p_j (each sphere containing just one p_j) and $H^2(\mathbb{R}^3 \setminus \{p_1, \dots, p_k\}, \mathbb{Z})$ is identified with the forms giving an integer value when integrated on the spheres. We give the spheres the orientation as boundary of balls.

Proposition 1 *(Gibbons-Hawking ansatz) Let M_0 be an S^1 bundle over $\mathbb{R}^3 \setminus \{p_1, \dots, p_k\}$ with $c_1(M_0)$ integrating -1 around the spheres containing just one of the p_j. Let $p = (\mu_1, \mu_2, \mu_3)$ be the projection. Let $i\alpha$ be a connection with*

$$d\alpha = p^*(-\star df) \tag{2.5}$$

where $f = a + \sum_{i=1}^{k} \frac{1}{2|x-p_i|}$ with $a \geq 0$. Then

$$g = (f \circ p)\sum_{i=1}^{3} d\mu_i^2 + (f^{-1} \circ p)\alpha \otimes \alpha \tag{2.6}$$

is a hyperkahler metric on M_0 which extends to a complete hyperkahler metric on $M = M_0 \sqcup \{\tilde{p_1}, \dots, \tilde{p_k}\}$

$\star$ is the Hodge star in $\mathbb{R}^3$ with the euclidean metric. We will not write pullbacks or composition with p when it is clear where things are. As follows from the proof p and the S^1 action extend smoothly to M with $p(\tilde{p_j}) = p_j$ and $\tilde{p_j}$ fixed by S^1

Remark 1 *The existence of M_0 follows from the bijection between isomorphism classes of S^1 bundles on a manifold X and $H^2(X, \mathbb{Z})$. The existence of α follows by picking any connection and then adding a suitable 1 form on the base.*

Because $\mathbb{R}^3 \setminus \{p_1, \dots, p_k\}$ is simply connected, any two connections with the same curvature are gauge equivalent, hence two different connections satisfying 2.5 define isometric metrics 2.6

Proof:

g is clearly a metric on M_0. Pick an orientation such that the volume element is

$$dV_g = f\alpha \wedge d\mu_1 \wedge d\mu_2 \wedge d\mu_3$$

Define the following 2 forms, where the subindices run in cyclic permutations from 1 to 3

$$\omega_i = \alpha \wedge d\mu_i + f d\mu_j \wedge d\mu_k \tag{2.7}$$

Then $\omega_i \wedge \omega_j = 2\delta_{ij} dV_g$ for every i, j and 2.5 is equivalent to the ω_i being closed.

Thinking of the ω_i as isomorphisms between TM_0 and T^*M_0 as $\omega_i(Y)(X) = \omega_i(X, Y)$ you can define the complex structures $I_i = \omega_k^{-1}\omega_j$. Then you check that these satisfy the quaternionic relations, that $g = \omega_i(., I_i.)$ and the I_i are integrable because $d\omega_i = 0$

We will proceed more explicitly. Let Y be the derivative of the S^1 action and $\frac{\tilde{\partial}}{\partial x_i}$ the horizontal lift of $\frac{\partial}{\partial x_i}$. Define

$$I_i \frac{\tilde{\partial}}{\partial x_i} = -fY$$

$$I_i \frac{\tilde{\partial}}{\partial x_j} = \frac{\tilde{\partial}}{\partial x_k}$$

Then $\omega_i = g(I_i., .)$, the I_i satisfy the quaternionic relations and are integrable because

$$[\frac{\tilde{\partial}}{\partial x_i} + ifY, \frac{\tilde{\partial}}{\partial x_j} - i\frac{\tilde{\partial}}{\partial x_k}] = (d\alpha(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) + \frac{\partial f}{\partial x_k})(-Y) + i(d\alpha(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_i}) + \frac{\partial f}{\partial x_j})(-Y) = 0$$

Hence we have a hyperkahler structure on M_0. Now we show that it has an extension to M.

For this assume that $p_i = 0$ and that there are no other p_j in the ball B centred at 0. Because of the condition on $c_1(M_0)$ we can find an S^1 equivariant map from $p^{-1}(B)$ to some ball centred at 0 in $\mathbb{R}^4$ with the S^1 action considered in the previous section. Using this map we have $p = (\frac{|w|^2 - |z|^2}{2}, izw)$ which extends smoothly mapping 0 to 0.

We write $f = \frac{1}{2|x|} + \tilde{f}$ where $\tilde{f}$ is smooth and harmonic on B so $-\star d\tilde{f} = d\tilde{\alpha}$ with $\tilde{\alpha}$ a smooth 1 form on B. Writting $\alpha = \alpha^0 + \tilde{\alpha}$. We have

$$\omega_i = \omega_i^0 + \tilde{\alpha} \wedge d\mu_i + \tilde{f} d\mu_j \wedge d\mu_k$$

where ω_i^0 are the symplectic forms on $\mathbb{R}^4$ of the previous section. From this expression for ω_i it follows that they extend smoothly to 0.

Now we prove that (M, g) is complete. Let $\{z_l\}$ be a Cauchy sequence on (M, g) assume $z_l \neq \tilde{p_j}$ for all l and all j from 1 to k. Then, because of the formula for g, $p(z_l)$ is bounded on $(\mathbb{R}^3 \setminus \{p_1, \dots, p_k\}, f|dx|^2)$. Assuming $p_1 = 0$ we have $f \geq \frac{1}{2|x|}$. So $p(z_l) = y_l$ is bounded on $(\mathbb{R}^3 \setminus \{0\}, \frac{1}{2|x|}|dx|^2)$.

Let $\gamma_l(t) = ty_l$ for $t \in [0, 1]$. Then

$$length(\gamma_l) = \frac{|y_l|}{(2|y_l|)^{1/2}} \int_0^1 t^{-1/2} dt = \sqrt{2|y_l|} \leq C$$

.

Hence $|y_l| \leq \frac{C^2}{2}$. Because the fibers of p are compact $\{z_l\}$ has a convergent subsequence.

$\square$

2.3 Asymptotic behavior

Let $G \subset U(2)$ be a finite subgroup acting freely on $\mathbb{C}^2 \setminus \{0\}$. The quotient $(\mathbb{C}^2 \setminus \{0\})/G$ is a Kahler manifold with metric h and complex structure I_0, both coming from the standard ones in $\mathbb{C}^2$.

In $\mathbb{C}^2$ use standard coordinates z, w and write $r = \sqrt{|z|^2 + |w|^2}$.

Definition 1 *A Kahler manifold (M, I, g) is said to be ALE if there exist $K \subset M$ compact and $\phi : M \setminus K \to (\mathbb{C}^2 \setminus B_R)/G$ a diffeomorphism such that*

$$|\nabla^k(\phi_* g - h)| = O(r^{-4-k}) \tag{2.8}$$

and

$$|\nabla^k(\phi_* I - I_0)| = O(r^{-4-k}) \tag{2.9}$$

for $k \geq 0$.

We claim that if M is given by the Gibbons-Hawking ansatz with

$$f = \sum_{j=1}^{k} \frac{1}{2|x - p_j|}$$

then (M, I, g) is ALE for any choice of complex structure I.
In this case $G = \mathbb{Z}_k$ acting on $\mathbb{C}^2$ by $e^{i2\pi/k}(z, w) = (e^{i2\pi/k}z, e^{-i2\pi/k}w)$.
To prove the claim we first describe $(\mathbb{C}^2/\mathbb{Z}_k, h)$ in a Gibbons-Hawking shape.
Denote by $\Pi : \mathbb{C}^2 \to \mathbb{C}^2/G$ the quotient projection and $\Pi(z, w) = [(z, w)]$. We have an $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ action on $(\mathbb{C}^2 \setminus \{0\})/\mathbb{Z}_k$ given by $e^{i\theta}[(z, w)] = [(e^{i\theta/k}z, e^{-i\theta/k}w)]$.
The map $\tilde{p} : \mathbb{C}^2 \to \mathbb{R}^3$

$$\tilde{p}(z, w) = (\frac{zw}{k}, \frac{|w|^2 - |z|^2}{2k})$$

Induces a map $p : (\mathbb{C}^2 \setminus \{0\})/\mathbb{Z}_k \to \mathbb{R}^3 \setminus \{0\}$.
This map p together with the S^1 action $e^{i\theta}[(z, w)] = [(e^{i\theta/k}z, e^{-i\theta/k}w)]$ give to $(\mathbb{C}^2 \setminus \{0\})/\mathbb{Z}_k$ the structure of an S^1 bundle over $\mathbb{R}^3 \setminus \{0\}$.
Denote by Y the derivative of this action. Then $\Pi_* \tilde{Y} = Y$ with

$$\tilde{Y} = \frac{1}{k}\left(\frac{\partial}{\partial \theta_1} - \frac{\partial}{\partial \theta_2}\right)$$

where $z = r_1 e^{i\theta_1}$ and $w = r_2 e^{i\theta_2}$.
The 1-form

$$\tilde{\alpha} = k \mathrm{Im} \frac{\overline{z}dz - \overline{w}dw}{|z|^2 + |w|^2}$$

projects to $(\mathbb{C}^2 \setminus \{0\})/\mathbb{Z}_k$, i.e $\tilde{\alpha} = \Pi^* \alpha$ with $\alpha(Y) = 1$.
On the other hand note that

$$\tilde{p}^*(|x|^2) = \frac{(|z|^2 + |w|^2)^2}{4k^2}$$

Now

$$|\tilde{Y}|^2 = \frac{|z|^2 + |w|^2}{k^2} = \tilde{p}^*(\frac{2|x|}{k}) = \tilde{p}^*(f^{-1})$$

where

$$f = \frac{k}{2|x|}$$

The claim is that

$$h = p^*(f g_{\mathbb{R}^3}) + (f^{-1} \circ p)\alpha^2 \tag{2.10}$$

To check this we compute $\Pi^* h$.
Recall that if $f_0 = \frac{1}{2|x|}$, $p_0 = (zw, \frac{|w|^2 - |z|^2}{2})$ and $\alpha_0 = \tilde{\alpha}/k$. Then

$$g_{\mathbb{R}^4} = p_0^*(f_0 g_{\mathbb{R}^3}) + (f_0^{-1} \circ p_0)\alpha_0^2$$

Denote by $m_{1/k}$ the multiplication by $1/k$ on $\mathbb{R}^3$ and note that $p \circ \Pi = \tilde{p} = m_{1/k} \circ p_0$. Then

$$\Pi^* h = k\tilde{p}^*(f_0 g_{\mathbb{R}^3}) + k^{-1}(f_0^{-1} \circ \tilde{p})k^2\alpha_0^2 = g_{\mathbb{R}^4}$$

since $\tilde{p}^*(f_0 g_{\mathbb{R}^3}) = p_0^* m_{1/k}^*(f_0 g_{\mathbb{R}^3}) = (1/k)p_0^*(f_0 g_{\mathbb{R}^3})$ and $f_0^{-1} \circ \tilde{p} = f_0^{-1} \circ m_{1/k} \circ p_0 = (1/k) f_0^{-1} \circ p_0$. Finally we note that

$$d\alpha = -p^*(\star df)$$

to check this we compute $\Pi^* d\alpha$ and use that $d\alpha_0 = -p_0^*(\star df_0)$

$$\Pi^* d\alpha = k d\alpha_0 = -p_0^*(\star df) = -\tilde{p}^* m_k^*(\star df) = -\tilde{p}^*(\star df) = \Pi^*(-p^* \star df)$$

Where we used that $m_k^*(\star df) = \star df$ since

$$m_k^*\left(\frac{x_i}{|x|^3} dx_j \wedge dx_k\right) = \frac{x_i}{|x|^3} dx_j \wedge dx_k$$

2.4 Constructing holomorphic functions

Let U be an open set in $\mathbb{R}^3 \setminus \{p_1, \ldots, p_k\}$. Assume U is the product of the x_1 axis with a simply connected domain in the $x_1 = 0$ plane. Consider the complex structure I on M given by $\frac{\partial}{\partial x_1}$.

In a trivialization of M_0 over U with coordinates (x_1, x_2, x_3, t) we have $\alpha = dt + \sum_{i=1}^3 a_i dx_i$. Where a_i are functions on U. If ϕ is a function from U to $\mathbb{R}$, we can take another trivialization with $\bar{t} = t + \phi$. Then $\alpha = d\bar{t} - d\phi + \sum_{i=1}^3 a_i dx_i$. Taking ϕ such that $\frac{\partial \phi}{\partial x_1} = a_1$ we assume $\alpha = dt + a_2 dx_2 + a_3 dx_3$. In this trivialization

$$Y = \frac{\partial}{\partial t}, \qquad \frac{\tilde{\partial}}{\partial x_1} = \frac{\partial}{\partial x_1}$$

$$\frac{\tilde{\partial}}{\partial x_2} = \frac{\partial}{\partial x_2} - a_2 \frac{\partial}{\partial t}, \qquad \frac{\tilde{\partial}}{\partial x_3} = \frac{\partial}{\partial x_3} - a_3 \frac{\partial}{\partial t}$$

.

The Cauchy-Riemann equations for a holomorphic function h are

$$\frac{\partial h}{\partial x_1} = if \frac{\partial h}{\partial t} \tag{2.11}$$

$$\frac{\partial h}{\partial x_2} + i \frac{\partial h}{\partial x_3} = (a_2 + i a_3) \frac{\partial h}{\partial t} \tag{2.12}$$

Because $d\alpha = - \star df$, we have

$$\frac{\partial f}{\partial x_1} = \frac{\partial a_2}{\partial x_3} - \frac{\partial a_3}{\partial x_2}$$

$$\frac{\partial f}{\partial x_2} = \frac{\partial a_3}{\partial x_1} \quad \text{and} \quad \frac{\partial f}{\partial x_3} = -\frac{\partial a_2}{\partial x_1}$$

.

From now on assume

$$\frac{\partial f}{\partial x_1} = 0 \qquad \text{on} \qquad \{x_1 = 0\}$$

So we have $\frac{\partial a_2}{\partial x_3} = \frac{\partial a_3}{\partial x_2}$ and we can do a gauge transformation depending just on x_2, x_3 and assume that $a_2 = a_3 = 0$ when $x_1 = 0$.

We look for h with weight 1 for the circle action, i.e. $h(e^{i\theta}.) = e^{i\theta} h(.)$. Then $\frac{\partial h}{\partial t} = ih$, $h = h(x_1, x_2, x_3, 0)e^{it}$. Then 2.11 becomes

$$\frac{\partial h}{\partial x_1} = -fh$$

so that

$$\frac{\partial \log h}{\partial x_1} = -f$$

.

And

$$h = e^{-u} h_0(x_2, x_3) e^{it}$$

where

$$u = \int_0^{x_1} f(s, x_2, x_3) ds$$

and $h_0(x_2, x_3) = h(0, x_2, x_3, 0)$. This h solves 2.11 for any choice of h_0.

Taking h_0 holomorphic in $x_2 + ix_3$ solves 2.12 in the slice $x_1 = 0$. In general you do not expect that h will solve 2.12 for other slices of x_1, but in our case the integrability condition (Bogomolony equation) gives

$$\frac{\partial h}{\partial x_2} + i\frac{\partial h}{\partial x_3} = e^{-u}(\frac{\partial h_0}{\partial x_2} + i\frac{\partial h_0}{\partial x_3})e^{it} + ih(\int_0^{x_1} -\frac{\partial f}{\partial x_3}(s, x_2, x_3)ds + i\int_0^{x_1} \frac{\partial f}{\partial x_2}(s, x_2, x_3)ds) =$$

$$= ih(\int_0^{x_1} \frac{\partial a_2}{\partial x_1}(s, x_2, x_3)ds + i\int_0^{x_1} \frac{\partial a_3}{\partial x_1}(s, x_2, x_3)ds) = ih(a_2 + ia_3)$$

Similarly

$$\hat{h} = e^{u} h_0 e^{-it}$$

is a holomorphic function with weight -1 for the S^1 action.

Notice that holomorphic functions in $x_2 + ix_3$ are holomorphic for I with weight 0, and that $h\hat{h} = h_0^2$

Example 1 *Consider*

$$f = 2c + \frac{1}{2|x|}$$

and the S^1 bundle given by

$$p(z, w) = \left(\frac{|w|^2 - |z|^2}{2}, zw\right)$$

with $e^{it}(z, w) = (e^{it}z, e^{-it}w)$ and connection form $i\alpha$ where

$$\alpha = Im\frac{\overline{z}dz - \overline{w}dw}{|z|^2 + |w|^2}$$

.

Let $U = \mathbb{R} \times (\mathbb{C} \setminus \{x_2 \leq 0\})$. Write $\xi = x_2 + ix_3 = re^{i\theta}$ with $-\pi < \theta < \pi$. Take the following trivialization

$$\Phi_U(x_1, \xi, t) = (z, w) = \left((\sqrt{|\xi|^2 + x_1^2} - x_1)^{1/2} e^{i\theta/2} e^{it}, (\sqrt{|\xi|^2 + x_1^2} + x_1)^{1/2} e^{i\theta/2} e^{-it}\right)$$

So $|z|^2 + |w|^2 = 2\sqrt{|\xi|^2 + x_1^2} = 2|x|$,

$$Im(\overline{z}dz) = (|x| - x_1)\frac{d\theta}{2} + (|x| + x_1)dt \quad \text{and} \quad Im(\overline{w}dw) = (|x| + x_1)\frac{d\theta}{2} - (|x| + x_1)dt$$

.

Hence

$$\alpha = dt - \frac{x_1}{2|x|}d\theta$$

And we are in the situation of $a_1 = 0$ *and* $a_2 = a_3 = 0$ *in* $x_1 = 0$. *Take* $h_0 = \sqrt{\xi}$. *Define* $h = ie^{-u}h_0e^{it}$ *and* $\hat{h} = e^u h_0 e^{-it}$. *Where*

$$u = \int_0^{x_1}(2c + \frac{1}{2\sqrt{s^2+|\xi|^2}})ds = 2cx_1 + \frac{1}{2}\log(x_1 + \sqrt{x_1^2+|\xi|^2}) - \frac{1}{2}\log|\xi|$$

So that

$$h = e^{c(|z|^2-|w|^2)}z \qquad \textit{and} \qquad \hat{h} = e^{c(|w|^2-|z|^2)}w$$

The map $\Psi = (h, \hat{h})$ *is a difeo of* $\mathbb{R}^4$ *which is holomorphic between* $\mathbb{R}^4$ *with the complex structure* I *and* $\mathbb{C}^2$ *in the dense set* $p^{-1}(U)$, *hence it is a biholomorphism.*

If γ is a 1 form, define $I\gamma(X) = \gamma(IX)$. Then $2i\partial\overline{\partial} = -dId$ on functions, where ∂ is the $\partial-$Operator defined by I.

In our case $Idx_1 = f^{-1}\alpha$ and $Idx_2 = -dx_3$. We want to find a Kahler potential for

$$\omega = \alpha \wedge dx_1 + f dx_2 \wedge dx_3$$

with $f = 2c + \frac{1}{2|x|}$ as in the example.

$$Id(|x|) = \frac{1}{|x|}(x_1 f^{-1}\alpha - x_2 dx_3 + x_3 dx_2)$$

Noting that $2df = d\frac{1}{|x|}$

$$dId(|x|) = \frac{1}{|x|}(d(x_1f^{-1}\alpha) - 2dx_2 \wedge dx_3) + 2df \wedge (x_1 f^{-1}\alpha - x_2dx_3 + x_3dx_2) \tag{2.13}$$

On the other hand

$$dId(x_1^2 + \frac{x_2^2+x_3^2}{2}) = 2d(x_1f^{-1}\alpha) - d(x_2dx_3 + x_3dx_2) = 2d(x_1f^{-1}\alpha) - 2dx_2 \wedge dx_3 \tag{2.14}$$

multypling 2.13 by 1/2, 2.14 by c and adding we obtain

$$dId\left(\frac{|x|}{2} + c(x_1^2 + \frac{x_2^2+x_3^2}{2})\right) =$$

$$fd(x_1f^{-1}\alpha) - (\frac{1}{|x|} + 2c)dx_2 \wedge dx_3 + x_1f^{-1}df \wedge \alpha - x_2 df \wedge dx_3 + x_3 df \wedge dx_2 =$$

$$= dx_1 \wedge \alpha - (\frac{1}{|x|} + 2c)dx_2 \wedge dx_3 + x_1 d\alpha - x_2 df \wedge dx_3 + x_3 df \wedge dx_2$$

Using $\frac{\partial f}{\partial x_i} = -\frac{x_i}{2|x|^3}$ and $d\alpha = -\star df$ we have $x_1 d\alpha - x_2 df \wedge dx_3 + x_3 df \wedge dx_2 = \frac{1}{2|x|}dx_2 \wedge dx_3$. So

$$dId(\frac{|x|}{2} + c(x_1^2 + \frac{x_2^2+x_3^2}{2})) = -\omega$$

or $i\partial\overline{\partial}(|x| + c(2x_1^2 + x_2^2 + x_3^2)) = \omega$. Using

$$|x| = \frac{|z|^2+|w|^2}{2} \qquad \text{and} \qquad 2x_1^2 + x_2^2 + x_3^2 = \frac{|z|^4+|w|^4}{2}$$

and the previous example we obtain the following complex description of the metric: on $\mathbb{C}^2$ with complex coordinates η_1, η_2. Define the functions l_1, l_2 implicitly by

$$e^{c(l_1^2-l_2^2)}l_1 = |\eta_1| \quad \text{and} \quad e^{c(l_2^2-l_1^2)}l_2 = |\eta_2|$$

.

Then

$$\omega = \frac{i}{2}\partial\overline{\partial}(l_1^2 + l_2^2 + c(l_1^4 + l_2^4))$$

is a complete hyperkahler metric with holomorphic volume form $d\eta_1 \wedge d\eta_2$. Moreover the function

$$\phi = l_1^2 + l_2^2 + c(l_1^4 + l_2^4)$$

satisfies

$$\det(\partial\overline{\partial}\phi) = 1 \quad \text{and} \quad i\partial\overline{\partial}\phi > 0$$

2.5 The manifold $zw = P(t)$

The Gibbons-Hawking ansatz produces an hyperkahler manifold (M, g) from a positive harmonic function on $\mathbb{R}^3 \backslash \{p_1 \dots p_k\}$. This manifold is Kahler with respect to a 2 sphere of complex structures. The objective of this section is to see which are the complex manifolds (M, I) that generically arise in this construction.

Let P be a polynomial in one complex variable with P and P' coprime, so $P(t) = (t - a_1) \dots (t - a_k)$ with $a_i \neq a_j$ if $i \neq j$. Let (z, w, t) be coordinates in $\mathbb{C}^3$ and consider the surface $X = \{zw = P(t)\}$. From

$$zdw + wdz = P'(t)dt$$

we see that

$$\Omega = \begin{cases} \frac{dz \wedge dw}{P'(t)} & \text{if } P'(t) \neq 0; \\ \frac{dt \wedge dw}{w} & \text{if } P(t) \neq 0. \end{cases}$$

is an holomorphic nowhere vanishing $(2, 0)$ form on X.

There is a $\mathbb{C}^*$ action on X preserving Ω given by $\lambda(z, w, t) = (\lambda z, \lambda^{-1}w, t)$ with derivative

$$Y = z\frac{\partial}{\partial z} - w\frac{\partial}{\partial w}$$

and $dt = i_Y \Omega$, so that t is an holomorphic moment map for this action.

There is an open cover of $X \setminus \{(0, 0, a_1) \dots (0, 0, a_k)\}$ by $U_1 = \{z \neq 0\}$ and $U_2 = \{w \neq 0\}$. Define charts $\phi_i = (u_i, v_i)$ from U_i to $\mathbb{C} \times \mathbb{C}^*$ where $u_1 = u_2 = t$ and $v_1 = z$, $v_2 = w^{-1}$. This maps are equivariant with respect to the $\mathbb{C}^*$ action on X and the $\mathbb{C}^*$ action $\lambda(u_i, v_i) = (u_i, \lambda v_i)$ on $\mathbb{C} \times \mathbb{C}^*$. In these coordinates

$$Y = v_i\frac{\partial}{\partial v_i} \quad \text{and} \quad \Omega = \frac{dv_i \wedge du_i}{v_i}$$

Finally the change of coordinates is the biholomorphic map of $\mathbb{C} \setminus \{a_1 \dots a_k\} \times \mathbb{C}^*$ given by

$$\phi_2 \circ \phi_1^{-1}(u, v) = \left(u, \frac{v}{(u - a_1) \dots (u - a_k)}\right)$$

Go back to the S^1 bundle M_0 over $\mathbb{R}^3 \setminus \{p_1 \dots p_k\}$. Consider the complex structure I coming from $\frac{\partial}{\partial x_1}$, asume $\xi = x_2 + ix_3$ is one to one on $\{p_1 \dots p_k\}$, this means that there is not segment joining two points and parallel to the x_1 axis. Write $a_i = \xi(p_i)$ and $L_i^+ = \{(s, a_i)$ with $s \geq x_1(p_i)\}$ and $L_i^- = \{(s, a_i)$ with $s \leq x_1(p_i)\}$.

The vector field $IY = f^{-1}\frac{\tilde{\partial}}{\partial x_1}$ is complete because the orbits are the horizontal lifts of the curves $(x_1(t), \xi_o)$ with $\frac{dx_1}{dt} = f^{-1}$ which go to ∞ in time $\infty = \int_{x_1}^{\infty} f(s, \xi_o)ds$. So we get a $\mathbb{C}^*$ action with derivative $\frac{Y - iIY}{2}$ and holomorphic moment map $H = \mu_2 + i\mu_3$ which is the orbit projection.

Consider the open cover of M_0 by $U_1 = M_0 \setminus p^{-1}(\{L_1^+ \dots L_k^+\})$ and $U_2 = M_0 \setminus p^{-1}(\{L_1^- \dots L_k^-\})$. Then H restricted to the U_i is a $\mathbb{C}^*$ bundle over $\mathbb{C}$, hence you can find biholomorphic maps $\phi_i = (u_i, v_i)$

which are $\mathbb{C}^*$ equivariant from U_i to $\mathbb{C}\times\mathbb{C}^*$, in such coordinates $H = u_i$, the derivative of the action is $v_i\frac{\partial}{\partial v_i}$ and $\Omega = \omega_2 + i\omega_3 = \frac{dv_i\wedge du_i}{v_i}$. Because the bundle is the same on $U_1\cap U_2$ the change of coordinates is a biholomorphism of $\mathbb{C}\setminus\{a_1\dots a_k\}\times\mathbb{C}^*$ given by $\phi_2\circ\phi_1^{-1}(u,v) = (u,\psi(u)v)$ where ψ is a holomorphic function from $\mathbb{C}\setminus\{a_1\dots a_k\}$ to $\mathbb{C}^*$.

In the model case $M_0 = \mathbb{C}^2\setminus\{0\}$, $H = zw$, $U_1 = \{z\neq 0\}$, $U_2 = \{w\neq 0\}$ and taking the trivializations given by the sections $(1,u)$ and $(u,1)$, we have $\psi(u) = u^{-1}$. In a neighbourhood of $\tilde{p_i}$ you can construct holomorphic functions with weights 1 and -1 as in the previous section, then locally we are in the model case, so ψ has a simple pole at a_i and we can write

$$\psi(u) = \frac{l(u)}{(u-a_1)\dots(u-a_k)}$$

where l is an holomorphic function from $\mathbb{C}$ to $\mathbb{C}^*$, changing the section s_1 giving the trivialization ϕ_1 by $l^{-1}s_1$ we can assume that $l = 1$ so we are in the situation considered at the beginning and we get a biholomorphism between M_0 and $X\setminus\{(0,0,a_1)\dots(0,0,a_k)\}$ which extends smoothly to M by sending $\tilde{p_i}$ to $(0,0,a_i)$, so we get a biholomorphism between M and X under which $t = \mu_2 + i\mu_3$ and $\omega_2 + i\omega_3 = \frac{dz\wedge dw}{P'(t)}$.

2.6 Computing the energy

On a Riemannian manifold (M,g) define the curvature operator on bivectors as

$$g\left(Rm(X\wedge Y), W\wedge Z\right) = R(X,Y,W,Z) = g\left(R(X,Y)Z, W\right)$$

where

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

We are interested in computing

$$E = \int_M |Rm|^2 dV_g$$

(the energy) for the manifolds (M,g) given by the Gibbons-Hawking ansatz.

First let us guess what the formula should be.

If (X^4,g) is a compact oriented Einstein 4 manifold with boundary then the Chern-Gauss-Bonnet theorem says that

$$8\pi^2\chi(X) = \int_X |Rm|^2 dV_g + \int_{\partial X}(sm)dA \tag{2.15}$$

where $\chi(X)$ is the Euler characteristic of X and (sm) is something that depends on the second fundamental form of ∂X inside X.

Applying the formula to a ball in $\mathbb{R}^4$, which has Euler characteristic 1, we see that $\int_{\partial X}(sm)dA = 8\pi^2$ in this case.

Consider (M,g) given by

$$f = \sum_{i=1}^k \frac{1}{2|x-p_i|}$$

then M is asymptotic to $\mathbb{R}^4/\mathbb{Z}^k$.

Taking X to be a large ball in M we have $\chi(X) = \chi(M) = k$ and we expect

$$\int_{\partial X}(sm)dA \to \frac{8\pi^2}{k} \tag{2.16}$$

as the radius of the ball goes to ∞. Puting 2.15 and 2.16 together we expect

$$E = 8\pi^2(k - \frac{1}{k})$$

In the following we will prove this by direct computation.

Let (M, g) be an oriented riemannian 4 manifold, let $\{e_0, e_1, e_2, e_3\}$ be an oriented orthonormal frame. Identify vectors with 1 forms using g. We want to find the components of Rm in the basis of $\Lambda^2 TM$ given by the selfdual and antiselfdual $\omega_i = e_0 \wedge e_i + e_j \wedge e_k$ and $\theta_i = e_0 \wedge e_i - e_j \wedge e_k$. Write

$$Rm = \begin{pmatrix} a_{ij} & b_{ij} \\ b_{ij}^t & c_{ij} \end{pmatrix}$$

where $2a_{ij} = g(Rm\omega_i, \omega_j)$, etc.

Let us concentrate on Λ^2_-.

It is a fact that there is a unique solution to the equations for 1 forms

$$d\theta_i = T_j \wedge \theta_k - T_k \wedge \theta_j$$

and the solution is given by

$$\nabla\theta_i = T_j \otimes \theta_k - T_k \otimes \theta_j$$

where ∇ is the induced Levi-Civita connection.

Once we compute the connection forms θ_i we can compute the curvature on $\Lambda^2_- T^*M$ as

$$R(X, Y)\theta_i = F_j(X \wedge Y)\theta_k - F_k(X \wedge Y)\theta_j$$

where $F_i = dT_i - T_j \wedge T_k$.

By easy computation

$$g(Rm(X \wedge Y), \theta_i) = -g(R(X, Y)\theta_j, \theta_k) = F_i(X \wedge Y)$$

So if we write $F_i^- = \sum_{j=1}^3 W_{ij}\theta_j$ then $2W_{ij} = g(Rm\theta_i, \theta_j) = 2c_{ij}$.

In our case $e_0 = f^{-1/2}\alpha$ and $e_i = f^{1/2}dx_i$. Then $d\omega_i = 0$ so the only non vanishing component of Rm is the c_{ij} Which we now compute.

$$d\theta_i = -2f_i dx_1 \wedge dx_2 \wedge dx_3$$

so you can check that

$$T_i = \frac{f_i}{f^2}\alpha - \frac{f_j dx_k - f_k dx_j}{f}$$

.

Computing $dT_i - T_j \wedge T_k$ you get that

$$c_{ij} = 3\frac{f_i f_j}{f^3} - \frac{f_{ij}}{f^2} - \delta_{ij}\frac{|Df|^2}{f^3}$$

which is $\frac{f}{2}$ times the trace free part of the Hessian of f^{-2}.

Now

$$|Rm|^2 = \sum_{i,j} c_{ij}^2 = 6f^{-6}|Df|^4 + \sum_{i,j} f^{-4}f_{ij}^2 - 6f^{-5}f_{ij}f_i f_j \tag{2.17}$$

On the other hand

$$\triangle f^{-1} = \sum_i (f^{-1})_{ii} = \sum_i (-f^{-2}f_i)_i = 2f^{-3}|Df|^2$$

We want to compute $\triangle\triangle f^{-1}$ so we keep on differentiating

$$(f^{-3}|Df|^2)_i = -3f^{-4}|Df|^2 f_i + 2\sum_j f^{-3}f_j f_{ij}$$

and

$$(f^{-3}|Df|^2)_{ii} = 12f^{-5}f_i^2|Df|^2 + \sum_j (-12)f^{-4}f_if_jf_{ij} + 2f^{-3}f_{ij}^2$$

So finally

$$\frac{1}{2}\triangle\triangle f^{-1} = 12f^{-5}|Df|^2 + \sum_{i,j} 2f^{-3}f_{ij}^2 - 12f^{-4}f_if_jf_{ij} \tag{2.18}$$

Comparing 2.17 and 2.18 we get the good looking formula

$$|Rm|^2 = \frac{1}{4}f^{-1}\triangle\triangle f^{-1} \tag{2.19}$$

Now we are ready to compute $E = \int |Rm|^2 dV_g$.
Note that the bundle projection map

$$p : (M, g) \to (\mathbb{R}^3, f|dx|^2)$$

is a riemannian submersion, the volume element of this last metric is $f^{3/2}dx_1 \wedge dx_2 \wedge dx_3$ and the length of the S^1 fiber over the point x is $2\pi f^{-1/2}(x)$ so we have

$$E = \frac{2\pi}{4}\int_{\mathbb{R}^3} f^{-1/2}f^{3/2}f^{-1}\triangle\triangle f^{-1}dx_1 \wedge dx_2 \wedge dx_3 = \frac{\pi}{2}\int_{\mathbb{R}^3} \triangle\triangle f^{-1} \tag{2.20}$$

By the divergence theorem

$$\int_{\mathbb{R}^3} \triangle\triangle f^{-1} = \lim_{R\to\infty}\int_{S_R} < D\triangle f^{-1}, n > dA - \sum_{i=1}^{k} \lim_{\epsilon\to 0}\int_{S_\epsilon(p_i)} < D\triangle f^{-1}, n > dA \tag{2.21}$$

Around the point p_i we can write $f^{-1} = 2|x - p_i| + h_i$ where h_i is smooth around p_i. Now

$$\triangle|x| = \sum_i (\frac{x_i}{|x|})_i = \sum_i \frac{1}{|x|} - \frac{x_i^2}{|x|^3} = \frac{2}{|x|}$$

and

$$< D\frac{1}{|x|}, n >= -\frac{1}{|x|^2}$$

so

$$\int_S < D\frac{1}{|x|}, n > dA = -4\pi$$

if S is any sphere around 0.
Then

$$\lim_{\epsilon\to 0}\int_{S_\epsilon(p_i)} < D\triangle f^{-1}, n > dA = 4\int_S < D\frac{1}{|x|}, n > dA = -16\pi \tag{2.22}$$

To compute the integral over the sphere that goes to ∞, let us say that a function h is in C_0 if D^kh is $O(|x|^{-k})$ as $|x| \to \infty$ for every $k \geq 0$ for example $h(x) = |x - p| - |x|$ is in C_0, for such functions

$$\lim_{R\to\infty}\int_{S_R} < D\triangle h, n > dA = 0$$

In the case that $f = \sum_{i=1}^k \frac{1}{2|x-p_i|}$, you can see (use example) that

$$f^{-1} = \frac{2|x|}{k} + h$$

with h in C_0. Then

$$\lim_{R\to\infty}\int_{S_R} < D\triangle f^{-1}, n > dA = \frac{-16\pi}{k} \tag{2.23}$$

Puting 2.20, 2.21, 2.22 and 2.23 together we get that

$$E = 8\pi^2(k - \frac{1}{k})$$

as we expected.
When (M, g) is given by

$$f = c + \sum_{j=1}^{k} \frac{1}{2|x - p_j|}$$

with $c > 0$, the computation above carries without change except that in this case $f^{-1} \in C_0$ so

$$\lim_{R\to\infty} \int_{S_R} < D\triangle f^{-1}, n > dA = 0 \tag{2.24}$$

And putting 2.20, 2.21, 2.22 and 2.24 together we get that

$$E = 8\pi^2 k$$

in this case.

3 Background metrics

Let $\{z = a_j w\}$ for $j = 1, \ldots, d-1$ and $\{w = 0\}$ be d lines in $\mathbb{C}^2$. We want to construct a flat Kahler metric with cone angle β along these lines. If $\beta > (d-2)/d$ there exist a metric $\overline{g}$ in $\mathbb{CP}^1 = \{[z : w]\}$ with constant curvature 4 and cone angle β on the points corresponding to the lines. More precisely, if $\xi = z/w$ we have $\overline{g} = e^{2\phi}|d\xi|^2$ with

$$u(\xi) = \phi(\xi) - (\beta - 1)\sum_{j=1}^{d-1} \log|\xi - a_j|$$

continuous on $\mathbb{C}$
and

$$\phi = (-\beta - 1)\log|\xi| + a$$

with $a(1/\eta)$ continuous at $\eta = 0$
Let $c = 2 + d\beta - d > 0$ and consider the 1-form

$$\alpha = \frac{i}{c}(\partial u - \overline{\partial} u) + dt$$

on $\mathbb{C} \times S^1$ with coordinates (ξ, e^{it}). The metric

$$g = \overline{g} + \frac{c^2}{4}\alpha^2$$

is locally isometric to $S^3(1)$.
Now we do the cone over g

$$h = d\rho^2 + \rho^2 g$$

We can define a complex structure I on this cone by

$$I\frac{\tilde{\partial}}{\partial x} = \frac{\tilde{\partial}}{\partial y}, \qquad I\frac{\partial}{\partial \rho} = \frac{2}{c\rho}\frac{\partial}{\partial t}$$

The fact is that (g, I) is a Kahler structure with Kahler form

$$\omega = \frac{i}{2}\partial\overline{\partial}\rho^2$$

Moreover the map

$$(0,\infty) \times \mathbb{C} \times S^1 \to \mathbb{C}^2 \setminus \{w=0\}$$

$$w = e^{u/c}\rho^{2/c}e^{it}, \qquad z = \xi w$$

is a biholomorphism.

From this we see that the metric that we were wanting to construct is the one with Kahler potential

$$\rho^2 = |w|^c e^{-u}$$

When $\beta = 1$ we are simply writting the euclidean metric as a cone over $S^3(1)$ and α is the Hopf connection. We start reviewing this and then we proceed to make clear all the assertions above.

3.1 Hopf map

Suppose you want to parametrize complex lines in $\mathbb{C}^2$. You can proceed directly and define the parameter space to be the manifold $\mathbb{CP}^1 = \{[z:w]\}$. Where $(z,w) \in \mathbb{C}^2 \setminus 0$ is identified with $(\lambda z, \lambda w)$ for $\lambda \in \mathbb{C}^*$.

This manifold is the union of the open set $U = \{[z:w], w \neq 0\}$ and the point $\infty = [1:0]$ which corresponds to the line $\{w=0\}$. On U we have the natural coordinate $\xi = z/w$ which parametrizes the line $\{z = \xi w\}$ that intersects the affine space $\{w=1\}$ at $(\xi, 1)$. Using ξ to identify U with $\mathbb{C}$ we have $\mathbb{CP}^1 = \mathbb{C} \sqcup \infty$.

On the other hand we have the map $p : \mathbb{C}^2 \to \mathbb{R}^3$

$$p(z,w) = (z\overline{w}, \frac{|w|^2 - |z|^2}{2})$$

This map is a bijection between complex lines in $\mathbb{C}^2$ and rays in $\mathbb{R}^3$, the parameter space of this last ones being clearly the 2 dimensional sphere. We denote $h = p|_{S^3(1)} : S^3(1) \subset \mathbb{C}^2 \to S^2(1/2) \subset \mathbb{R}^3$. Note that

$$p(\xi w, w) = |w|^2 \left(\xi, \frac{1-|\xi|^2}{2}\right) = |w|^2(1+|\xi|^2)\left(\frac{\xi}{1+|\xi|^2}, \frac{1-|\xi|^2}{2(1+|\xi|^2)}\right)$$

So p maps the line $\{z = \xi w\}$ to the ray cutting $S^2(1/2)$ at $q = \left(\frac{\xi}{1+|\xi|^2}, \frac{1-|\xi|^2}{2(1+|\xi|^2)}\right)$ and the line $\{w=0\}$ to the ray passing through the south pole $S = (0, -1/2)$.

Stereographic projection from $S^2(1/2) \setminus S \to \mathbb{C}$ is given by

$$(\eta, x_0) \to \frac{2\eta}{2x_0 + 1}$$

The image of q under this map is ξ. Finally we get a map from $\mathbb{CP}^1$ to $S^2(1/2)$ sending $[1:0] \to S$ and points in U with coordinate ξ to points in $S^2(1/2) \setminus S$ using the inverse of stereographic projection.

Use the normal coordinates $(0,\pi/2) \times S^1 \to S^2(1/2) \subset \mathbb{R}^3$

$$(r, e^{i\theta}) \to \left(\frac{\sin(2r)}{2}e^{i\theta}, \frac{\cos(2r)}{2}\right)$$

in this coordinates the induced metric from $\mathbb{R}^3$ is

$$g_{S^2(1/2)} = dr^2 + \frac{\sin^2(2r)}{4}d\theta^2$$

.

Composing this map with stereographic projection we get the map $(0,\pi/2) \times S^1 \to \mathbb{C}$

$$(r, e^{i\theta}) \to \frac{\sin(2r)}{\cos(2r)+1}e^{i\theta} = \tan(r)e^{i\theta} = \xi$$

and in this coordinates the metric is

$$g_{S^2(1/2)} = \frac{|d\xi|^2}{(1+|\xi|^2)^2}$$

Now we want to write the euclidean metric $g_{\mathbb{C}^2} = |dz|^2 + |dw|^2$ as a cone over $S^3(1)$. Write $z = r_1 e^{i\theta_1}$ and $w = r_2 e^{i\theta_2}$ with $r_1, r_2 > 0$. Let $\rho \in (0,\infty)$ and $r \in (0, \pi/2)$ be given by

$$r_1 = \rho \sin(r)$$

$$r_2 = \rho \cos(r)$$

Then

$$\begin{aligned} g_{\mathbb{C}^2} = dr_1^2 + r_1^2 d\theta_1^2 + dr_2^2 + r_2^2 d\theta_2^2 = \\ (\sin(r)d\rho + \rho\cos(r)dr)^2 + \rho^2\sin^2(r)d\theta_1^2 + (\cos(r)d\rho - \rho\sin(r)dr)^2 + \rho^2\cos^2(r)d\theta_2^2 \\ = d\rho^2 + \rho^2\left(dr^2 + \sin^2(r)d\theta_1^2 + \cos^2(r)d\theta_2^2\right) \end{aligned}$$

The pullback of the euclidean metric under the map $(0,\pi/2)\times S^1 \times S^1 \to S^3(1) \subset \mathbb{C}^2$

$$(r, e^{i\theta_1}, e^{i\theta_2}) \to (\sin(r)e^{i\theta_1}, \cos(r)e^{i\theta_2})$$

is

$$g_{S^3(1)} = dr^2 + \sin^2(r)d\theta_1^2 + \cos^2(r)d\theta_2^2$$

Recall the Hopf map $h(z,w) = (z\overline{w}, (|w|^2 - |z|^2)/2)$. In the coordinates $(r, e^{i\theta_1}, e^{i\theta_2})$ for $S^3(1)$ and $(r, e^{i\theta})$ for $S^2(1/2)$ introduced above

$$z\overline{w} = \sin(r)\cos(r)e^{i(\theta_1-\theta_2)} = \frac{\sin(2r)}{2}e^{i(\theta_1-\theta_2)}$$

and

$$\frac{|w|^2-|z|^2}{2} = \frac{\cos^2(r)-\sin^2(r)}{2} = \frac{\cos(2r)}{2}$$

so that

$$h(r, e^{i\theta_1}, e^{i\theta_2}) = (r, e^{i\theta})$$

where $\theta = \theta_1 - \theta_2$

The Hopf map together with the action of S^1 on $S^3(1)$, $e^{is}(z,w) = (e^{is}z, e^{is}w)$, is an S^1 bundle over S^2. In our coordinates, the derivative of the S^1 action is

$$Y = \frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2}$$

The distribution given by the orthogonal complements to Y with respect to $g_{S^3(1)}$ is a connection with connection form given by

$$\alpha = g_{S^3(1)}(Y,.) = \sin^2(r)d\theta_1 + \cos^2(r)d\theta_2$$

And we have the nice fact that

$$h^*g_{S^2(1/2)} + \alpha^2 = dr^2 + \sin^2(r)\cos^2(r)(d\theta_1 - d\theta_2)^2 + (\sin^2(r)d\theta_1 + \cos^2(r)d\theta_2)^2 = dr^2 + \sin^2(r)d\theta_1^2 + \cos^2(r)d\theta_2^2$$

i.e,

$$g_{S^3(1)} = h^*g_{S^2(1/2)} + \alpha^2$$

which in particular implies that h is a Riemannian submersion between $S^3(1)$ and $S^2(1/2)$.

The curvature of the connection is

$$d\alpha = 2\sin(r)\cos(r)dr\wedge d\theta_1 - 2\cos(r)\sin(r)dr\wedge d\theta_2 = h^*(\sin(2r)dr\wedge d\theta) = h^*(\frac{1}{2}K_{S^2(1/2)}dV_{S^2(1/2)})$$

Where $K_{S^2(1/2)} = 4$ and

$$dV_{S^2(1/2)} = \frac{\sin(2r)}{2}dr\wedge d\theta$$

are the curvature and volume form of $S^2(1/2)$.
The map

$$(r, e^{i\theta_1}, e^{i\theta_2}) \to (r, e^{i\theta}, e^{it})$$

where $\theta = \theta_1 - \theta_2$ and $t = \theta_2$. Is S^1 equivariant with respect to the actions $e^{is}(r, e^{i\theta_1}, e^{i\theta_2}) = (r, e^{i(\theta_1+s)}, e^{(i\theta_2+s)})$ and $e^{is}(r, e^{i\theta}, e^{it}) = (r, e^{i\theta}, e^{i(t+s)})$. I.e. it is a trivialization of the bundle.
In this trivialization

$$\alpha = dt + \sin^2(r)d\theta = dt + \alpha_0$$

In the coordinate $\xi = \tan(r)e^{i\theta}$, using that

$$\sin^2(r) = \frac{\tan^2(r)}{1+\tan^2(r)}$$

we have

$$\alpha_0 = \frac{|\xi|^2}{1+|\xi|^2}d\theta$$

.

Now let us keep track of the different coordinates. From a point (ρ, ξ, e^{it}) in $(0,\infty)\times\mathbb{C}\times S^1$ with $\xi = |\xi|e^{i\theta}$ we go to $(\rho, r = \arctan(|\xi|), e^{i\theta}, e^{it})$ then to $(\rho, r, e^{i(\theta+t)} = e^{i\theta_1}, e^{it} = e^{i\theta_2})$ and finally to the point $(z,w)\in\mathbb{C}^2$ with

$$z = \rho\sin(r)e^{i\theta_1} = \rho\frac{|\xi|}{\sqrt{1+|\xi|^2}}e^{i\theta}e^{it} = \rho\frac{\xi}{\sqrt{1+|\xi|^2}}e^{it} = \rho w$$

$$w = \rho\cos(r)e^{i\theta_2} = \rho\frac{1}{\sqrt{1+|\xi|^2}}e^{it}$$

Note that keeping ξ fixed and moving ρ and t we obtain the line $\{z = \xi w\}$, the description we started with. Resuming

$$(0,\infty)\times\mathbb{C}\times S^1 \to \mathbb{C}^2\setminus\{w=0\}$$

$$(\rho, \xi, e^{it}) \to (\xi w, w)$$

$$w = \rho e^{it}\frac{1}{\sqrt{1+|\xi|^2}}$$

$$|dz|^2 + |dw|^2 = d\rho^2 + \rho^2\left(\frac{1}{(1+|\xi|^2)^2}|d\xi|^2 + (dt + \frac{|\xi|^2}{1+|\xi|^2}d\theta)^2\right)$$

3.2 Cone metrics on S^2

Let $0 < \beta < 1$ and consider the region in the plane delimited by two rays forming an angle of $2\pi\beta$. Gluing these rays you get a cone which we can describe, using polar coordinates, as a metric on $\mathbb{R}^2 \setminus \{0\}$

$$g_0 = dr^2 + \beta^2 r^2 d\theta^2$$

This metric gives to $\mathbb{R}^2 \setminus \{0\}$ the complex structure of $\mathbb{C}^*$. Indeed if $\xi = r^{1/\beta} e^{i\theta}$ then $r = |\xi|^\beta$, so that

$$dr = \beta |\xi|^{\beta - 1} d|\xi|$$

and

$$dr^2 + \beta^2 r^2 d\theta^2 = \beta^2 |\xi|^{2\beta - 2} |d\xi|^2$$

Let S be a Riemann surface, $\beta > 0$ and $p \in S$. Let g be a compatible metric smooth on $S \setminus \{p\}$. We say that g has cone angle β at p if we can find complex coordinates with $\xi(p) = 0$ in which

$$g = e^{2\phi} |d\xi|^2$$

with

$$\phi = \phi_0 + (\beta - 1) \log |\xi|$$

and ϕ_0 admits a *continuous* extension to 0. The definition of a metric with cone angles β_j at points p_j is straightforward.

Recall that the curvature of the metric $g = e^{2\phi}|d\xi|^2$ is given by the formula

$$K = -e^{-2\phi} \triangle \phi$$

where $\triangle = \partial^2/\partial x^2 + \partial^2/\partial y^2$ and $\xi = x + iy$.

In the above situation we have the following

Lemma 1 *If K extends continuously to 0 then*

$$\lim_{\xi \to 0} |\xi| \left| \frac{\partial \phi_0}{\partial \overline{\xi}} \right| = 0$$

Proof:

□

Let g is a metric on a compact Riemann surface S with cone angles β_j at $p_j \in S$, $j = 1, \ldots, d$ and curvature K_g extending continuously to S. Write $g = e^{2\phi} g_0$ with g_0 smooth.

The Gauss-Bonnet theorem tell us that

$$\frac{1}{2\pi} \int_S K_{g_0} dV_{g_0} = \chi(S)$$

where $\chi(S)$ is the Euler characteristic of S.

Outside $\{p_j\}$ we have

$$K_g dV_g = K_{g_0} dV_{g_0} - 2i \partial \overline{\partial} \phi$$

Integrating this over $S \setminus \cup_{j=1}^{d} B_\epsilon(p_j)$ and using Stokes' theorem

$$\int_S K_g dV_g = \int_S K_{g_0} dV_{g_0} + 2i \sum_{j=1}^{d} \lim_{\epsilon \to 0} \int_{S_\epsilon(p_j)} \overline{\partial} \phi \tag{3.1}$$

Around a point $p_j = 0$ we have $\phi = (\beta_j - 1) \log |\xi| + \phi_0$, with ϕ_0 as in the lemma, so

$$\lim_{\epsilon \to 0} \int_{S_\epsilon(0)} \overline{\partial} \phi = (\beta_j - 1) \lim_{\epsilon \to 0} \int_{S_\epsilon(0)} \frac{\partial}{\partial \overline{\xi}} \log |\xi| d\overline{\xi} = (\beta_j - 1) \lim_{\epsilon \to 0} \int_{S_\epsilon(0)} \frac{1}{2\overline{\xi}} d\overline{\xi} = -\pi i (\beta_j - 1)$$

Inserting this limit in equation 3.1 we get the Gauss- Bonnet formula in this context

$$\frac{1}{2\pi}\int_S K_g dV_g = \chi(S) + \sum_{j=1}^{d}(\beta_j - 1) \tag{3.2}$$

We are interested in metrics with $K_g = const > 0$ and $0 < \beta_j = \beta < 1$ for all j. In this case the right hand side of 3.2 has to be positive, in particular $\chi(S) > 0$, so S is the Riemann sphere and

$$\beta > \frac{d-2}{d} \tag{3.3}$$

Conversely we have the following

Theorem 1 *Let $a_1, \ldots, a_{d-1} \in \mathbb{C}$ and $\beta > (d-2)/d$. Then there exist ϕ smooth on $\mathbb{C} \setminus \{a_1, \ldots, a_{d-1}\}$ such that*

$$\triangle\phi = -4e^{2\phi}$$

$$\phi - (\beta - 1)\sum_{j=1}^{d-1}\log|\xi - a_j|$$

is continuous on $\mathbb{C}$ and

$$\phi = (-\beta - 1)\log|\xi| + a$$

with $a(1/\eta)$ continuous at $\eta = 0$

The theorem says that $g = e^{2\phi}|d\xi|^2$ has constant curvature 4 and cone angle β at a_j. The last condition means that if $\eta = 1/\xi$ then

$$|d\xi|^2 = |\eta|^{-4}|d\eta|^2$$

so that

$$g = |\xi|^{-2\beta-2}e^{2a}|\eta|^{-4}|d\eta|^2 = |\eta|^{2\beta-2}e^{2a}|d\eta|^2$$

and g has cone angle β at $\eta = 0$. In other words g is a metric on $\mathbb{CP}^1 = \{[z : w]\}$ with coordinates $\xi = z/w$ and $\eta = w/z$. The metric has cone angle β at the points p_j with coordinate $\xi(p_j) = a_j$ for $j = 1, \ldots d-1$ and $p_d = [1 : 0]$.

In the section on the Hopf map we saw that the map

$$(0, \frac{\pi}{2}) \times S^1 \to \mathbb{C}$$

which sends $(r, e^{i\theta})$ to $\xi = \tan(r)e^{i\theta}$ pulls back the metric on $S^2(1/2)$

$$\frac{1}{(1+|\xi|^2)^2}|d\xi|^2$$

to

$$dr^2 + \frac{\sin^2(2r)}{4}d\theta^2$$

Now it is tempting to consider

$$g = dr^2 + \beta^2\frac{\sin^2(2r)}{4}d\theta^2$$

If we are lucky we are going to make the metric conformal by writting $\xi = |\xi|e^{i\theta}$ with $|\xi| = (\tan(r))^{1/\beta}$. I.e, $r = \arctan(|\xi|^\beta)$, so that

$$dr = \frac{\beta|\xi|^{\beta-1}}{1+|\xi|^{2\beta}}d|\xi|$$

Using that

$$\frac{\sin^2(2r)}{4} = \sin^2(r)\cos^2(r) = \frac{\tan^2(r)}{(1+\tan^2(r))^2}$$

We get that

$$dr^2 + \beta^2 \frac{\sin^2(2r)}{4} d\theta^2 = \beta^2 \frac{|\xi|^{2\beta-2}}{(1+|\xi|^{2\beta})^2}|d\xi|^2$$

So

$$g = \beta^2 \frac{|\xi|^{2\beta-2}}{(1+|\xi|^{2\beta})^2}|d\xi|^2 \tag{3.4}$$

is our candidate for a cone metric on the sphere with constant curvature 4.
If $\eta = 1/\xi$

$$g = \beta^2 \frac{|\eta|^{-2\beta+2}}{(1+|\eta|^{-2\beta})^2}|\eta|^{-4}|d\eta|^2 = \beta^2 \frac{|\eta|^{2\beta-2}}{(1+|\eta|^{2\beta})^2}|d\eta|^2$$

So g is a metric on $\mathbb{CP}^1$ with angle β at $[0:1]$ and $[1:0]$. Writing $g = e^{2\phi}|d\xi|^2$ we get

$$\phi = \log\beta + (\beta-1)\log|\xi| - \log(1+|\xi|^{2\beta})$$

Computing

$$\frac{\partial\phi}{\partial\overline{\xi}} = \frac{\beta-1}{2\overline{\xi}} - \frac{\beta}{1+|\xi|^{2\beta}}\xi^\beta\overline{\xi}^{\beta-1}$$

$$\frac{\partial^2\phi}{\partial\xi\partial\overline{\xi}} = -\beta^2\frac{|\xi|^{2\beta-2}}{1+|\xi|^{2\beta}} + \beta^2\frac{|\xi|^{4\beta-2}}{(1+|\xi|^{2\beta})^2} = -\beta^2\frac{|\xi|^{2\beta-2}}{(1+|\xi|^{2\beta})^2}$$

Because $4\frac{\partial^2}{\partial\xi\partial\overline{\xi}} = \triangle$ we have that $\triangle\phi = -4e^{2\phi}$, i.e, g has constant curvature 4.
We have that

$$\phi - (\beta-1)\log|\xi| = \log\frac{\beta}{1+|\xi|^{2\beta}}$$

is continuous on $\mathbb{C}$. And that

$$a = \phi + (1+\beta)\log|\xi| = \log\frac{\beta|\xi|^{\beta-1}|\xi|^{\beta+1}}{1+|\xi|^{2\beta}} = \log\frac{\beta|\xi|^{2\beta}}{1+|\xi|^{2\beta}}$$

so

$$a(1/\eta) = \log\frac{\beta|\eta|^{-2\beta}}{1+|\eta|^{-2\beta}} = \log\frac{\beta}{1+|\eta|^{2\beta}}$$

is continuous at $\eta = 0$

3.3 Cone metrics on $\mathbb{C}^2$

The objective of this section is to prove the following

Proposition 2 *If $l_1,\dots,l_d$ are different complex lines in $\mathbb{C}^2$ and $\frac{d-2}{d} < \beta < 1$. Then there exist a Kahler metric on $\mathbb{C}^2\setminus\{0\}$ which is flat and has cone singularities of angle β along the given lines.*

Let $a_1,\dots,a_{d-1}\in\mathbb{C}$ and

$$\frac{d-2}{d} < \beta < 1$$

.

Let ϕ be a smooth real function on $\mathbb{C}\setminus\{a_1,\dots,a_{d-1}\}$ solution of

$$-e^{-2\phi}\triangle\phi = 4$$

such that

$$u = \phi - (\beta - 1)\sum_{j=1}^{d-1}\log|\xi - a_j|$$

is continuous on $\mathbb{C}$ and

$$a = \phi + (1+\beta)\log|\xi|$$

is continuous at ∞, i.e, $a(1/\eta)$ is continuous at $\eta = 0$

Then $h = e^{2\phi}|d\xi|^2$ is a metric with constant curvature $K_h = 4$ and cone angle β at $a_1, \dots, a_{d-1}, \infty$. Let

$$c = 2 + d\beta - d = \int_{\mathbb{C}} K_h dV_h > 0$$

$dV_h = e^{2\phi}dx \wedge dy$, where $\xi = x + iy$, is the volume element of h. Define a smooth real 1-form on $\mathbb{C} \setminus \{a_1, \dots, a_{k-1}\}$ by

$$\alpha_0 = \frac{i}{c}(\partial u - \overline{\partial} u)$$

Then

$$d\alpha_0 = -\frac{1}{c}2i\partial\overline{\partial}u = \frac{1}{c}(-\triangle u)dx \wedge dy = \frac{1}{c}K_h dV_h$$

so that $\int_{\mathbb{C}} d\alpha_0 = 1$. Note that because

$$\lim_{\xi \to a_j}|\xi - a_j|\left|\frac{\partial u}{\partial\overline{\xi}}\right| = 0$$

we have that

$$\lim_{\epsilon\to 0}\int_{S_\epsilon(a_j)}\alpha_0 = 0$$

for $j = 1, \dots, d-1$.

Because $H^1_{dR}(\mathbb{C}\setminus\{a_1, \dots, a_{k-1}\})$ is generated by $d\theta_j$ where (r_j, θ_j) are polar coordinates on $\mathbb{C} \setminus \{a_j\}$ we have that any other 1-form $\tilde{\alpha_0}$ on $\mathbb{C}\setminus\{a_1, \dots, a_{k-1}\}$ satisfying

$$d\tilde{\alpha_0} = \frac{1}{c}K_h dV_h \tag{3.5}$$

$$\lim_{\epsilon\to 0}\int_{S_\epsilon(a_j)}\tilde{\alpha_0} = 0 \tag{3.6}$$

differs from α_0 by the differential of a function, i.e, the two conditions determine α_0 up to gauge.

On $\mathbb{C}\setminus\{a_1, \dots, a_{d-1}\} \times S^1$ with coordinates (ξ, e^{it}) we define the metric

$$g = h + \frac{c^2}{4}(dt + \alpha_0)^2$$

Note that if $\tilde{\alpha_0}$ satisfies 3 and 4.2 and we define $\tilde{g} = h + (c^2/4)(dt + \tilde{\alpha_0})^2$. Then $\alpha_0 = \tilde{\alpha_0} + df$ for some function f. The map $F(\xi, e^{it}) = (\xi, e^{i(t+f)})$ does $F^*\tilde{g} = g$.

We call $\alpha = \alpha_0 + dt$ and think of it as a connection on $\mathbb{C}\setminus\{a_1, \dots, a_{d-1}\}\times S^1$. In this way of thinking F is a gauge transformation.

We claim that g is locally isometric to $S^3(1)$. If $p \in \mathbb{C}\setminus\{a_1, \dots, a_{d-1}\}$ we know, by the uniformization theorem for riemannian manifolds, that there exist a map Φ from a disc around 0 to a neighborhood of p such that

$$\Phi^* h = dr^2 + \frac{\sin^2(2r)}{4} d\theta^2$$

$$d\Phi^* \alpha_0 = \Phi^* d\alpha_0 = \Phi^*(\frac{1}{c} K_h dV_h) = \frac{2}{c}\sin(2r) dr \wedge d\theta$$

Composing (Φ, Id) with a gauge transformation we can assume $\alpha_0 = (2/c)\sin^2(r)d\theta$. So g in this coordinates will be

$$dr^2 + \frac{\sin^2(2r)}{4} d\theta^2 + (\frac{c}{2}dt + \sin^2(r)d\theta)^2$$

If we restrict to $t \in (-\pi, \pi) \subset S^1$, say, and define $\overline{t} = (c/2)t$, then $g = dr^2 + \frac{\sin^2(2r)}{4} d\theta^2 + (d\overline{t} + \sin^2(r)d\theta)^2$. Which is the expression for $g_{S^3(1)}$ that we found in the section on the Hopf map.

Now we form the cone over g

$$\overline{g} = d\rho^2 + \rho^2 g$$

On $U = (0, \infty) \times \mathbb{C} \setminus \{a_1, \ldots, a_{d-1}\} \times S^1$ with coordinates (ρ, ξ, e^{it}). Because g is locally $S^3(1)$ we have that $\overline{g}$ is locally the euclidean metric on $\mathbb{R}^4$. Now we define an almost complex structure compatible with g by

$$I\frac{\tilde{\partial}}{\partial x} = \frac{\tilde{\partial}}{\partial y}$$

$$I\frac{\partial}{\partial \rho} = \frac{2}{c\rho}\frac{\partial}{\partial t}$$

where

$$\frac{\tilde{\partial}}{\partial x} = \frac{\partial}{\partial x} - \alpha\left(\frac{\partial}{\partial x}\right)\frac{\partial}{\partial t}$$

$$\frac{\tilde{\partial}}{\partial y} = \frac{\partial}{\partial y} - \alpha\left(\frac{\partial}{\partial y}\right)\frac{\partial}{\partial t}$$

are the horizontal lifts of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

Note that $\{d\rho, \alpha, dx, dy\}$ are point wise linearly independent 1-forms with dual frame $\{\frac{\partial}{\partial \rho}, \frac{\partial}{\partial t}, \frac{\tilde{\partial}}{\partial x}, \frac{\tilde{\partial}}{\partial y}\}$. We define

$$\omega = \overline{g}(I., .) = \rho^2 e^{2\phi} dx \wedge dy + \frac{c\rho}{2} d\rho \wedge \alpha$$

Then

$$d\omega = 2\rho e^{2\phi} d\rho \wedge dx \wedge dy - \frac{c\rho}{2} d\rho \wedge \left(\frac{4}{c} e^{2\phi} dx \wedge dy\right) = 0$$

It is easy to see that

$$\left[\frac{\tilde{\partial}}{\partial x} + i\frac{\tilde{\partial}}{\partial y}, \frac{\partial}{\partial \rho} + i\frac{2}{c\rho}\frac{\partial}{\partial t}\right] = 0$$

Thus $(\overline{g}, \omega, I)$ is a Kahler structure.

There is a nice potential for ω

$$Id(\rho^2) = 2\rho I d\rho = -c\rho^2 \alpha$$

$$d(-c\rho^2\alpha) = -2c\rho d\rho \wedge \alpha - 4\rho^2 e^{2\phi} dx \wedge dy$$

From $2i\partial\overline{\partial} = -dId$ we get

$$\omega = \frac{i}{2}\partial\overline{\partial}\rho^2$$

Now we want to see that I has an extension to $\mathbb{C}^2$. For doing this we will find holomorphic functions z and w with weight 1 for the S^1 action.

The Cauchy-Riemann equations for a holomorphic function h are

$$\frac{\partial h}{\partial \rho} + i\frac{2}{c\rho}\frac{\partial h}{\partial t} = 0$$

$$\frac{\partial h}{\partial x} + i\frac{\partial h}{\partial y} = \alpha\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\frac{\partial h}{\partial t}$$

When $\frac{\partial h}{\partial t} = ih$ we have

$$\frac{\partial h}{\partial \rho} = \frac{2}{c\rho}h$$

$$\frac{\partial h}{\partial \overline{\xi}} = i\alpha\left(\frac{\partial}{\partial \overline{\xi}}\right)h$$

$\rho^{2/c}$ solves the first equation. Set $h = \rho^{2/c}e^{it}h_0$ with $h_0 = h_0(\xi)$ and recall that $\alpha = dt + \frac{i}{c}(\partial u - \overline{\partial}u)$. The second equation gives

$$\frac{\partial \log h_0}{\partial \overline{\xi}} = \frac{1}{c}\frac{\partial u}{\partial \overline{\xi}}$$

So we can take $h_0 = e^{u/c}$ (or h_0 multiplied by a constant) . Note that ξ is an holomorphic function with weight 0. Finally we set

$$z = \xi w$$

$$w = e^{u/c}\rho^{2/c}e^{it}$$

As a check

$$\rho^2 = |w|^c e^{-u}$$

If we start with the metric with two cone singularities on S^2

$$h = \beta^2\frac{|\xi|^{2\beta-2}}{(1+|\xi|^{2\beta})^2}|d\xi|^2 = e^{2\phi}|d\xi|^2$$

$$\phi = \log\beta + (\beta-1)\log|\xi| - \log(1+|\xi|^{2\beta})$$

We can take

$$u = -\log(1+|\xi|^{2\beta})$$

From $|\xi| = |z|/|w|$ and $c = 2 + d\beta - d = 2\beta$ we get

$$\rho^2 = |w|^c e^{-u} = |w|^{2\beta}\left(1 + \frac{|z|^{2\beta}}{|w|^{2\beta}}\right) = |z|^{2\beta} + |w|^{2\beta}$$

is the potential for $\overline{g}$

4 Examples

4.1 Quotients of Gibbons-Hawking metrics

Let M be given by the Gibbons-Hawking ansatz with

$$f = \sum_{j=1}^{k} \frac{1}{2|x - p_j|} \qquad \text{where } p_j = (0, a_j)$$

and a_j are the k roots of -1.

We constructed three holomorphic functions (z, w, t) with weights $(1, -1, 0)$ mapping (M, I) biholomorphicaly to $zw = t^k + 1$, where I is the complex structure given by $\frac{\partial}{\partial x_1}$.

Let ϕ in SO(3) be the rotation of angle $2\pi/k$ in the x_2, x_3 plane and fixing the x_1 axis. Pulling back M_0 by ϕ gives a bundle isomorphic to M_0, hence we get a S^1 equivariant map A of M_0 covering ϕ. Because $f \circ \phi = f$ and ϕ is in SO(3), $\phi^* \star df = \star df = -d\alpha$. Then $A^* d\alpha = d\alpha$ so $A^*\alpha - \alpha = dh$ with h a smooth real function on $\mathbb{R}^3 \setminus \{p_1, \ldots, p_k\}$. Define $\Phi = A \circ e^{-ih}$, this is a S^1 equivariant map covering ϕ and $\Phi^*\alpha = \alpha$. Note that adding a constant to h changes Φ by a constant gauge transformation.

If X is a vector field on $\mathbb{R}^3$ and $\tilde{X}$ denotes the horizontal lift, then $\Phi_* \tilde{X} = \widetilde{\phi_* X}$. From this, $\Phi_* Y = Y$ and the definition of I it follows that Φ preserves I. You can use Hartago's theorem to extend Φ to M by doing the corresponding permutation of the $\tilde{p_j}$. From $g = f|dx|^2 + f^{-1}\alpha \otimes \alpha$ it is clear that Φ is an isometry.

Remember that $t = x_2 + ix_3$, so $t \circ \Phi = e^{i2\pi/k} t$. Choose Φ fixing the fiber over 0, then it will fix the points over $\{0\} \times \mathbb{R}$. Assume $z \circ \Phi = z$ and $w \circ \Phi = w$, so our isometry is $\Phi(z, w, t) = (z, w, e^{i2\pi/k} t)$. Then we can pushforward the metric on M to $\mathbb{C}^2$ via the map $(z, w, t) \to (z, w)$ and obtain a Ricci flat Kahler metric with cone angle $\beta = 1/k$ along the curve $zw = 1$.

In the quotient you still have an S^1 action $e^{i\theta}(z, w) = (e^{i\theta} z, e^{-i\theta} w)$ preserving the Kahler structure. Locally you can pushforward the holomorphic 2 form

$$\frac{dz \wedge dw}{kt^{k-1}} \to \frac{dz \wedge dw}{k(zw - 1)^{1-1/k}} = \beta(zw - 1)^{\beta - 1} dz \wedge dw$$

the S^1 action preserves this locally defined 2 form, and there is a locally defined Hamiltonian

$$t = (zw - 1)^{1/k}$$

The model at infinite for this metric should be $\mathbb{C}_\beta \times \mathbb{C}_\beta$.

This metric in $\mathbb{C}^2$ is the quotient of M by a cyclic subgroup of isometries of order k, hence the energy of this metric is $1/k$ times the energy of M, we saw that $E(M) = 8\pi^2(k - 1/k)$. Then the energy of the quotient is

$$E = 8\pi^2(1 - 1/k^2) = 8\pi^2(1 - \beta^2)$$

Taking $k = l.m$ you get an ALE metric on $X_1 = \{zw = t^{lm} + 1\}$ invariant under $(z, w, t) \to (z, w, e^{i2\pi/lm} t)$. Let $X_2 = \{zw = t^m + 1\}$, then $(z, w, t) \to (z, w, t^l)$ is a covering branched over $D = \{(z, w, 0)/zw = 1\} \subset X_2$. Pushing forward the metric on X_1 you get an ALE metric on X_2 with cone angle $\beta = 1/l$ along D and energy

$$E = 8\pi^2 l^{-1}(lm - 1/lm) = 8\pi^2(m - \beta^2/m)$$

In this case the model at infinite should be $(\mathbb{C}_\beta \times \mathbb{C}_\beta)/\mathbb{Z}_m$.

On the other hand, starting with a smooth curve $C = \{P(z, w) = 0\} \subset \mathbb{C}^2$. Consider $X = \{P(z, w) = t^k\} \subset \mathbb{C}^3$. The map $(z, w, t) \to (z, w)$ is a covering from X to $\mathbb{C}^2$ branched over C. If you can find an ALE metric on X invariant under $(z, w, t) \to (z, w, e^{i2\pi/k})$ then you can pushforward the metric and obtain an ALE metric on $\mathbb{C}^2$ with cone angle $1/k$ along C.

4.2 Quotients of D_4 gravitational instantons

Let D_4 be the subgroup of $SU(2)$ generated

$$(z, w) \to (iz, -iw)$$

and

$$(z, w) \to (w, -z)$$

The polynomials

$$u = \frac{i}{2}(z^4 + w^4)$$
$$v = z^2 w^2$$

and

$$t = \frac{1}{2}(z^5 w - w^5 z)$$

are invariant under the action of D_4 and satisfy $t^2 + u^2 v + v^3 = 0$. Moreover

$$\mathbb{C}^2/D_4 \cong \{(u, v, t) \in \mathbb{C}^3 \quad \text{s.t.} \quad t^2 + u^2 v + v^3 = 0\}$$

It is well know that there exist hyperkahler ALE metrics on deformations of the singularity as for example $X = \{t^2 + u^2 v + v^3 = 0 + 1\}$. If the metric is invariant under $(u, v, t) \to (u, v - t)$ then we can take the quotient to obtain a metric on $\mathbb{C}^2$ with cone angle $\beta = 1/2$ along a cubic with three asymptotic lines $\{v = 0\}$, $\{u + iv = 0\}$ and $\{u - iv = 0\}$.

Next we do this with more detail.

Let $H \subset SU(2)$ be a finite subgroup. We associate a simply laced Dynking diagram Γ to H. Points of Γ correspond to non-identity conjugacy classes of H.

As in Lie groups we associate to Γ a set of roots Δ which spans an inner product vector space $\mathfrak{h}$ with real dimension equal to the number of points in Γ.

We have actions of the Weyl group W and of the automorphisms of the graph $\mathrm{Aut}(\Gamma)$ on $\mathfrak{h}$ which combine to give an action of $W \ltimes \mathrm{Aut}(\Gamma)$ on $\mathfrak{h}$.

Let X be the smooth manifold underlying the crepant resolution of $\mathbb{C}^2/H$. It turns out that the deformations of the singularity are diffeomorphic to X.

There is a vector space isomorphism $H^2_{dR}(X, \mathbb{R}) \cong \mathfrak{h}$. Under this isomorphism the set Δ correspond to the Poincare duals of the spheres in $H_2(X, \mathbb{Z})$ with self intersection -2.

Consider now $H \subset G \subset U(2)$ where G is finite and H is a normal subgroup of G. Let $K = G/H$ and note that K acts on $\mathbb{C}^2/H$.

We have a group morphism $\psi : K \to \mathrm{Aut}(\Gamma)$ which sends gH to the automorphism $C_h \to (gH)C_h(gH)^{-1}$ where C_h is a conjugacy class in H.

Let $\chi : K \to W \ltimes \mathrm{Aut}(\Gamma)$ a group morphism such that $\pi \circ \chi = \psi$. Where $\pi : W \ltimes \mathrm{Aut}(\Gamma) \to \mathrm{Aut}(\Gamma)$ is the projection.

Theorem 2 *Let $H \subset G \subset U(2)$ and χ as above. Assume that $(\alpha, \beta) \in H^2(X, \mathbb{R}) \times H^2(X, \mathbb{C})$ satisfy that for every $\delta \in H_2(X, \mathbb{Z})$ with $\delta^2 = -2$ either $\alpha(\delta) \neq 0$ or $\beta(\delta) \neq 0$ and the additional conditions*

$$\chi(gH)\alpha = \alpha \tag{4.1}$$

$$\chi(gH)\beta = \det(gH)\beta \tag{4.2}$$

Then there exist on X an hyperkahler ALE structure asymptotic to $\mathbb{C}^2/H$ with Kahler forms $\omega_1, \omega_2, \omega_3$ satisfying $[\omega_1] = \alpha$, $[\Omega = \omega_2 + i\omega_3] = \beta$.

There also exist an action of K on X by isometries Φ_{gH} which is asymptotic to the action of K on $\mathbb{C}^2/H$ and

$$\Phi^*_{gH}\omega_1 = \omega_1$$

$$\Phi_{gH}^*\Omega = \det(gH)\Omega$$

The action of Φ_{gH} on $H^2(X,\mathbb{R})$ is given by $\chi(gH)$.

Now consider $H \subset SU(2)$ generated by $(z,w) \to (iz,-iw)$ and $(z,w) \to (-w,z)$. This is a group of order eight, this group projects under the covering $SU(2) \to SO(3)$ to a group of order four isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by $diag(1,-1,-1)$ and $diag(-1,1,-1)$.

The polynomials

$$u = \frac{i}{2}(z^4 + w^4), \qquad v = z^2w^2, \qquad t = \frac{1}{2}(z^5w - w^5z)$$

are invariant under the action of H and give an isomorphism

$$\mathbb{C}^2/H \cong \{(u,v,t) \in \mathbb{C}^3 \quad \text{s.t.} \quad t^2 + vu^2 + v^3 = 0\}$$

as complex spaces.

Let G be the subgroup of $U(2)$ generated by H and $(z,w) \to (w,z)$. Note that $K = G/H \cong \mathbb{Z}_2$ acts on $\mathbb{C}^2/H$ by $(u,v,t) \to (u,v,-t)$ so that (u,v) give an isomorphism of complex manifolds $\mathbb{C}^2/G \cong \mathbb{C}^2$.

Because $G \subset U(2)$ we can pushforward the metric to obtain a new metric on $\mathbb{C}^2$ with coordinates (u,v) and a cone angle of $\beta = 1/2$ along the image of the fixed points by K which occur when $t = 0$, i.e, when $vu^2 + v^3 = v(u+iv)(u-iv) = 0$ which is the union of three lines.

The Dynkin diagram Γ corresponding to H is D_4 with $\mathrm{Aut}(\Gamma) \cong \mathbb{Z}_3$ so that

$$\psi : \mathbb{Z}_2 \cong K \to \mathrm{Aut}(\Gamma) \cong \mathbb{Z}_3$$

is trivial.

$\mathfrak{h}$ is $\mathbb{R}^4$ with orthonormal basis $\epsilon_1, \dots, \epsilon_4$. $\Delta = \{\pm(\epsilon_i \pm \epsilon_j), i \neq j\}$ is the set of roots. The Weyl group is generated by permutations of the elements of the basis and even number of sign changes so that $W \cong \mathbb{Z}_2^3 \ltimes S_4$. For us the important thing is that there exist $a \in W$ with $a^2 = 1$ and acting on $\mathfrak{h}$ as multiplication by -1.

We define $\chi : K \to W \ltimes \mathrm{Aut}(\Gamma)$ by sending the generator of K to a.

The conditions 3 and 4.2 in the theorem above for $(\alpha,\beta) \in H^2(X,\mathbb{R}) \times H^2(X,\mathbb{C})$ are

$$\alpha = -\alpha$$

$$\beta = \beta$$

so $\alpha = 0$ and any choice of β solve this. If we pick the real or imaginary part of β in one of the Weyl chambers we fulfill the requirement $\beta(\delta) \neq 0$ for any $\delta \in H_2(X,\mathbb{Z})$ with $\delta^2 = -2$ so we have a freedom to choose β of four complex parameters.

$\alpha = 0$ implies that (X, I_1) is affine. The goal is to identify (X, I_1) with $\{(u,v,t) \in \mathbb{C}^3 \quad \text{s.t.} \quad t^2 + vu^2 + v^3 = 1\}$,say, and K acting as $(u,v,t) \to (u,v,-t)$. So that X/K is $\mathbb{C}^2$ with coordinates u, v and the metric has cone angle $1/2$ along $\{vu^2 + v^3 = 1\}$.

4.3 The Green's function of a wedge

Let U be a domain in $\mathbb{R}^n$, $n \geq 3$, consider the Neumann problem

$$\begin{cases} -\triangle u = f & \text{in } U; \\ \frac{\partial u}{\partial n} = g & \text{in } \partial U. \end{cases}$$

Suppose that U is bounded and ∂U is C^1. Fix x in U with $B(x,\epsilon) \subset U$. Let $V = U \setminus B(x,\epsilon)$ and apply the Green's identity

$$\int_V \triangle u v - \triangle v u = \int_{\partial V} \frac{\partial u}{\partial n} v - \frac{\partial v}{\partial n} u$$

with

$$v = \Phi(y-x) = \frac{1}{(n-2)\gamma_{n-1}|y-x|^{n-2}}$$

where γ_{n-1} is the area of S^{n-1}. Letting $\epsilon \to 0$ you get the identity

$$u(x) = -\int_U \Phi(y-x)\triangle u(y)dy + \int_{\partial U} \Phi(y-x)\frac{\partial u}{\partial n}(y) - u(y)\frac{\partial \Phi}{\partial n}(y-x)u(y)dS(y) \tag{4.3}$$

The upshot to solve the Neumann problem is to get rid of the last term by solving

$$\begin{cases} \triangle \phi_x = 0 & \text{in } U; \\ \frac{\partial \phi_x}{\partial n} = -\frac{\partial \Phi}{\partial n}(y-x) & \text{in } \partial U. \end{cases}$$

Then

$$\int_U \phi_x \triangle u = \int_{\partial U} u(y)\frac{\partial \Phi}{\partial n}(y-x) + \phi_x(y)\frac{\partial u}{\partial n}(y)dS(y) \tag{4.4}$$

Combining 4.3 and 4.4 we get that if u in $C^2(\overline{U})$ solves the Neumann problem, then it is given by

$$u(x) = \int_U G(x,y)f(y)dy + \int_{\partial U} G(x,y)g(y)dS(y) \tag{4.5}$$

where $G(x,y) = \Phi(y-x) + \phi_x(y)$ is a solution of the equation

$$\begin{cases} -\triangle_y G(x,.) = \delta_x & \text{in } U; \\ \frac{\partial G(x,.)}{\partial n} = 0 & \text{in } \partial U. \end{cases}$$

We call this G the Green's function for U (for the Neumann problem).

Remove the negative real axis from $\mathbb{C}$ write $z = re^{i\tilde{\theta}}$ with $-\pi < \tilde{\theta} < \pi$, let $W = \{z : -\pi/k < \tilde{\theta} < \pi/k\}$ and $U = \mathbb{R} \times W \subset \mathbb{R}^3$, let x be in U and

$$G(x,y) = \sum_{j=1}^{k} \frac{1}{4\pi|y-p_j|}$$

where the p_j are obtained from $p_1 = x$ by reflecting it across the planes $\mathbb{R} \times \{\tilde{\theta} = \pm\pi/k\}$.

This G satisfies the distributional equation of the Green's function for U, as you can notice by thinking of the p_j as unit charges and seeing that the normal component electric field is zero on $\mathbb{R} \times \{\tilde{\theta} = \pm\pi/k\}$.

The function f of the last section is, up to a factor and rotation, $G(x,.)$ for this U and $x = (0,1)$ and the construction of the last section can be interpreted as doing the Gibbons-Hawking ansatz to $\overline{U}$ with $f(y) = 2\pi G(y,x)$.

You obtain an hyperkahler 4 manifold with boundary. Then you glue the parts of the boundary corresponding to the fibers over $\mathbb{R} \times \{\tilde{\theta} = \pi/k\}$ and $\mathbb{R} \times \{\tilde{\theta} = -\pi/k\}$ by lifting the isometry $(x_1, x_2, x_3) \to (x_1, -x_2, x_3)$. You get a cone angle of $2\pi/k$ along the preimage of $\mathbb{R} \times \{0\}$.

This approach allow us to generalize the construction for all $0 < \beta < 1$, so let us do it.

$W = \{z : -\pi\beta < \tilde{\theta} < \pi\beta\}$ and $U = W \times \mathbb{R} \subset \mathbb{R}^3$. The map $(r, \theta, x_1) \to (r, \tilde{\theta}, x_1)$ where $\beta\theta = \tilde{\theta}$. Is an isometry between $\mathbb{C}_\beta \times \mathbb{R}$ minus the negative real axis and U with the euclidean metric.

If f is a smooth function of compact support and you want to find a solution of $-\triangle_g u = f$ with $g = dr^2 + \beta^2 r^2 d\theta^2 + dx_1^2$ you can pushforward f to $\tilde{f}$ in U solve $-\triangle \tilde{u} = \tilde{f}$ in U and $\frac{\partial \tilde{u}}{\partial n} = 0$ in ∂U and then pullback $\tilde{u}$.

It is a fact that there exist a locally integrable function $G(x,y)$, smooth away from the diagonal and x, y in $S = \{0\} \times \mathbb{R}$ such that

$$u(x) = \int G(x,y)f(y)dV_g(y)$$

is a weak solution in H_0^1 of $-\triangle_g u = f$ for every f smooth of compact support.

G is positive and harmonic with respect to g. In the case that $\beta = 1/k$ G is the pullback of

$$G(x,y) = \sum_{j=1}^{k} \frac{1}{4\pi|y-p_j|}$$

as before.

Let (X^3, g) is a flat manifold, f is a smooth positive function with $\triangle_g f = 0$ and $[\star_g df] \in H^2(M, \mathbb{Z})$. Take an S^1 bundle with connection $i\alpha$ with $d\alpha = -\star_g df$, define

$$g = fp^*g + f^{-1}\alpha \otimes \alpha$$

This g is locally hyperkahler because locally we can arrange coordinates in which the metric is the euclidean one and define the forms ω_j.

Now apply this to $(\mathbb{C}_\beta \times \mathbb{R}) \setminus S$ and $f(y) = 2\pi G(x,y)$ with $x = (1,0)$. Because in this case X is not simply connected we can add to α a closed 1 form in M and get a different metric (unless the form is exact). Looking at the examples with $\beta = 1/k$ we see that we should choice α such that the holonomy of the connection goes to zero when we consider small loops arorund S.

In the total space of the S^1 bundle, M_0, you can define an integrable complex structure I by sending the horizontal lift of $\frac{\partial}{\partial x_1}$ to $-fY$ and the horizontal lift of $\frac{\partial}{\partial r}$ to the horizontal lift of $\frac{1}{\beta r}\frac{\partial}{\partial \theta}$. With this I, g becomes Kahler.

We can choose a trivialization of M_0 over $(\mathbb{C} \setminus \{z \in \mathbb{R}, z \leq 1\}) \times \mathbb{R}$ such that $\frac{\partial}{\partial x_1}$ is parallel and the bundle is flat over $\{x_1 = 0\}$. In this trivialization we can write holomorphic functions with weights 1 and -1 as

$$h_\pm = e^{\mp u} h_0 e^{\pm it}$$

where

$$u = \int_0^{x_1} f(x_2, x_3, s) ds$$

and

$$\frac{\partial h_0}{\partial r} + i \frac{1}{\beta r} \frac{\partial h_0}{\partial \theta} = 0$$

M_0 has a natural smooth extension $\tilde{M_0}$ as an S^1 bundle over $\mathbb{R}^3 \setminus \{x\}$.

Choosing

$$h_0 = (1 - r^c e^{i\theta})^{1/2}$$

where $c = 1/\beta$, does that (h_+, h_-) extend as a difeo (use that $\int_0^\infty f(x_2, x_3, s) ds = \infty$) between $\tilde{M_0}$ and $\mathbb{C}^2 \setminus \{0\}$ which holomorphic on M_0.

Let $M = \tilde{M_0} \sqcup \tilde{x}$ with $\tilde{x}$ projecting to x. Then we can extend I to M and M becomes biholomorphic to $\mathbb{C}^2$ the metric will have a cone singularity of angle β along $zw = h_0^2 = 1$ (this is when $r = 0$).

Doing this construction with

$$f(y) = 2\pi \sum_{j=1}^{k} G(x_j, y)$$

you get Ricci flat metrics on ALE spaces with cone angle β along a curve.

4.4 The energy

Let us compute the energy.

We have $g = dr^2 + \beta^2 r^2 d\theta^2 + dx_1^2$ and $f(y) = 2\pi G(x,y)$ where $x = (1,0)$.

The energy is given by

$$E = \frac{\pi}{2} \int_{\mathbb{R}^3} \triangle_g \triangle_g f^{-1} dV_g$$

.

Integrating by parts

$$\frac{2}{\pi}E = \lim_{R\to\infty}\int_{C_R} g(D_g\triangle_g f^{-1}, n)dA_g - \lim_{\epsilon\to 0}\int_{C_\epsilon} g(D_g\triangle_g f^{-1}, n)dA_g - \lim_{\epsilon\to 0}\int_{S_\epsilon} g(D_g\triangle_g f^{-1}, n)dA_g \quad (4.6)$$

. Where C_R is a cylinder of radius R, C_ϵ is a cilinder of radius ϵ around S and S_ϵ is a sphere of radius ϵ around x.

Because g is isometric to the euclidean metric in a ball containing S_ϵ we have

$$\lim_{\epsilon\to 0}\int_{S_\epsilon} g(D_g\triangle_g f^{-1}, n)dA_g = -16\pi \quad (4.7)$$

as in the euclidean case.

Note that

$$G(0,y) = \frac{1}{4\pi\beta|y|}$$

From the scaling $G(x,y) = \lambda G(\lambda x, \lambda y)$ and the fact that there is a constant c s.t.

$$|G(w_1,z) - G(w_2,z)| \le c|w_1 - w_2|^{1/\beta} \qquad \text{for all } |z| = 1 \text{ and } |w_1|, |w_2| \le 1/2$$

we have

$$|f(y) - G(0,y)| = |y|^{-1}\left|G(\frac{x}{|y|}, \frac{y}{|y|}) - G(0, \frac{y}{|y|})\right| \le c|y|^{-1-1/\beta} \qquad \text{when } |y| \ge 2$$

Replacing f^{-1} by $2\beta|y|$ and noting that $dA_g = \beta dA$ with dA the euclidean area. We get

$$\lim_{R\to\infty}\int_{C_R} g(D_g\triangle_g f^{-1}, n)dA_g = -\beta^2 16\pi \quad (4.8)$$

Finally $\triangle_g f^{-1} = O(|Df|_g^2)$, using that f is β smooth we have $|Df|_g^2 = O(r^{2\beta^{-1}-2})$ as $r \to 0$.

Then $g(\nabla_g\triangle_g f^{-1}, n) = O(r^{2\beta^{-1}-3})$. Because $dA_g = \beta r d\theta \wedge dx_1$ we have

$$\int_{C_\epsilon} g(D_g\triangle_g f^{-1}, n)dA_g = O(\epsilon^{2\beta^{-1}-2}) \to 0 \qquad \text{as } \epsilon \to 0 \quad (4.9)$$

From 4.6, 4.7, 4.8 and 4.9 we get

$$E = 8\pi^2(1-\beta^2) \quad (4.10)$$

Similarly you can do the construction using $f(y) = 2\pi\sum_{j=1}^k G(p_j, y)$. When you compute the energy the contribution small balls around the p_j will be the euclidean one and the contrbution from the large cylinder will be β^2 times the corresponding smooth ALE case, so the energy will be

$$E = 8\pi^2(k - \beta^2/k)$$

.

5 Analytic Approach

5.1 Joyce's work

On $\mathbb{C}^n$ with standard coordinates $z_1, \dots, z_n$ write $r = \sqrt{\sum_{j=1}^n |z_j|^2}$. A smooth real function ρ is called a radius function if $\rho \ge 1$ and $\rho = r$ when $r \ge 2$.

We define C_γ^∞ to be the space of smooth real functions u for which there exists constants C_k with $k = 0, 1, 2, \dots$ such that

$$|\partial^\alpha u| \le C_k \rho^{\gamma-k}$$

where $k = |\alpha|$ is the order of the derivative.

Write $\omega = \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\overline{z_j}$ for the standard Kahler form. The main result of this section is the following

Theorem 3 *Let $\gamma \in (-2n,-2)$ and $f \in C^\infty_\gamma$. Then there exist a unique $\phi \in C^\infty_{\gamma+2}$ such that*

$$(\omega + i\partial\overline{\partial}\phi)^n = e^f \omega^n \tag{5.1}$$

Let

$$g_{\alpha\overline{\beta}} = \delta_{\alpha\beta} + \frac{\partial^2 \phi}{\partial z_\alpha \partial \overline{z_\beta}}$$

then 5.1 reads $\det(g_{\alpha\overline{\beta}}) = e^f$. It follows that the eigenvalues of the matrix $(g_{\alpha\overline{\beta}})$ are never zero. Because the second derivatives of ϕ decay to zero, the eigenvalues of $(g_{\alpha\overline{\beta}})$ are all positive for $r >> 1$. By continuity it follows that they are everywhere positive. Hence $\omega_\phi = \omega + i\partial\overline{\partial}\phi$ defines a Kahler metric on $\mathbb{C}^n$ with Ricci form $-i\partial\overline{\partial}f$.

The uniqueness of solutions of 5.1 is quite elementary and we proceed to the proof.

Proposition 3 *Let $\phi_1, \phi_2 \in C^\infty_\gamma$ for some $\gamma < 0$ satisfy*

$$(\omega + i\partial\overline{\partial}\phi_1)^n = (\omega + i\partial\overline{\partial}\phi_2)^n = e^f\omega^n$$

Then $\phi_1 = \phi_2$

Proof:

Denote by $\omega_1 = \omega + i\partial\overline{\partial}\phi_1$ and $\omega_2 = \omega + i\partial\overline{\partial}\phi_2$ the corresponding Kahler forms. Let $\phi = \phi_2 - \phi_1 \in C^\infty_\gamma$. Then $\omega_2 = \omega_1 + i\partial\overline{\partial}\phi$ and

$$0 = \omega_2^n - \omega_1^n = \frac{1}{2} dd^c\phi \wedge (\omega_2^{n-1} + \omega_2^{n-2}\wedge\omega_1 + \ldots + \omega_1^{n-1})$$

Let B_R be the ball of radius R and $p \ge 1$, using Stokes' theorem

$$\int_{B_R} d\left(|\phi|^p d^c\phi \wedge (\omega_2^{n-1} + \ldots + \omega_1^{n-1})\right) = \int_{\partial B_R} |\phi|^p d^c\phi \wedge (\omega_2^{n-1} + \ldots + \omega_1^{n-1})$$

Taking $p > \frac{2-2n}{\gamma}$ the r.h.s. integral goes to 0 as $R \to \infty$.
Expanding the l.h.s. and using

$$d\phi \wedge d^c\phi \wedge \omega_2^{n-1} = \frac{1}{n}|\nabla\phi|_2^2 \omega_2^n$$

$$d\phi \wedge d^c\phi \wedge \omega_2^{n-1-j} \wedge \omega_1^j = F_j \omega_2^n$$

where $|\nabla\phi|_2$ refers to the norm of the gradient of ϕ computed with the metric given by ω_2 and F_j are non negative functions for $j = 1, \ldots, n-1$

We get

$$\int_{\mathbb{C}^n} |\phi|^{p-1}|\nabla\phi|_2^2\omega_2^n \le 0$$

from which we deduce that ϕ is constant, and since it decays to 0 that $\phi = 0$.

□

Remark 2 *We have to use that ϕ_1 and ϕ_2 decay to 0. As an application of the Gibbons-Hawking ansatz we will give an example of a real function u on $\mathbb{C}^2$ such that $i\partial\overline{\partial}u > 0$, $\det(\partial\overline{\partial}u) = 1$ and u does not differ by any constant with $|z_1|^2 + |z_2|^2$.*

Let $\gamma+2<0$ and $U\subset C^{\infty}_{\gamma+2}$ be the open subset of functions ϕ such that $(\omega+i\partial\overline{\partial}\phi)^n$ is a positive multiple of ω^n. Define a function $f=F(\phi)$ by

$$f=\log\frac{(\omega+i\partial\overline{\partial}\phi)^n}{\omega^n}$$

Expanding

$$(\omega+i\partial\overline{\partial}\phi)^n=\sum_{j=0}^{n}\binom{n}{j}\omega^{n-j}\wedge(i\partial\overline{\partial}\phi)^j$$

and using $\log(1+x)=1+x+O(x^2)$ for small x we see that $f\in C^{\infty}_{\gamma}$. Theorem 3 asserts that $F:U\to C^{\infty}_{\gamma}$ is a diffeomorphism.

Let $\phi\in U$ and $\psi\in C^{\infty}_{\gamma+2}$ a tangent vector. Let $\omega_\phi=\omega+i\partial\overline{\partial}\phi$. Using that $i\partial\overline{\partial}\psi\wedge\omega_\phi^{n-1}=\frac{1}{n}(\triangle_\phi\psi)\omega_\phi^n$ and expanding as above we get that $DF_\phi(\psi)=\triangle_\phi\psi$. Where $\triangle_\phi$ is $-1/2$ of the Hodge laplacian defined by ω_ϕ.

Of course we have not been formal with the topology on the spaces C^{∞}_{γ}. But to show that F maps $C^{*}_{\gamma+2}$ to C^{*}_{γ} and that $DF_\phi=\triangle_\phi$ is the same in the upcoming spaces in which we will be formal with the topology.

The existence of a solution to 5.1 is proved using the continuity method by solving

$$(\omega+i\partial\overline{\partial}\phi_t)^n=e^{tf}\omega^n$$

starting at $t=0$ with $\phi_0=0$.

The way to prove openess of the set of t for which there exist a solution ϕ_t is to find adequate Banach spaces in which the linearization of the equation is an isomorphism and then apply the implicit function theorem. In the next section we develop the linear theory necessary for this. These results are also used to prove regularity of solutions.

5.2 Schauder Estimates

Consider the model metric

$$g=\beta^2|z|^{2\beta-2}|dz|^2+\sum_{i=1}^{n-1}|dw_i|^2$$

on $\mathbb{C}^n$ with complex coordinates $(z,w_1,\dots,w_{n-1})$. The laplacian of this metric is given by

$$\triangle_\Omega=\frac{|z|^{2-2\beta}}{\beta^2}4\frac{\partial^2}{\partial z\partial\overline{z}}+\sum_{i=1}^{n-1}4\frac{\partial^2}{\partial w_i\partial\overline{w_i}}$$

We want to find Banach spaces for which $\triangle_\Omega$ is an isomorphism. Note that $\triangle_\Omega f=<i\partial\overline{\partial}f,\Omega>$ where

$$\Omega=\frac{i}{2}(\beta^2|z|^{2\beta-2}dz\wedge d\overline{z}+\sum_{i=1}^{n-1}dw_i\wedge d\overline{w_i})$$

is the Kahler form of g.

Let $\xi=|z|^{\beta-1}z=re^{i\theta}$ and $\eta=e^{i\theta}(dr+i\beta rd\theta)$, then $2^{-1/2}\{\eta,dw_i\}$ is an orthonormal trivialization of $\Omega^{1,0}$ with dual frame $2^{1/2}\{\frac{|z|^{1-\beta}}{\beta}\frac{\partial}{\partial z},\frac{\partial}{\partial w_i}\}$.

The components of ∂f with respect to $\{\eta,dw_i\}$ are given by $\beta^{-1}|z|^{1-\beta}\frac{\partial f}{\partial z},\frac{\partial f}{\partial w_i}$.

The components of $i\partial\overline{\partial}f$ with respect to $\{\eta\wedge\overline{\eta},\eta\wedge d\overline{w_i},dw_i\wedge\overline{\eta},dw_i\wedge d\overline{w_j}\}$ are respectively

$$|z|^{2-2\beta}\frac{\partial^2 f}{\partial z\partial\overline{z}},\qquad |z|^{1-\beta}\frac{\partial^2 f}{\partial z\partial\overline{w_i}},\qquad |z|^{1-\beta}\frac{\partial^2 f}{\partial w_i\partial\overline{z}},\qquad \frac{\partial^2 f}{\partial w_i\partial\overline{w_j}}$$

up to some multiples of β.

Let $\Phi:\mathbb{C}^n\to\mathbb{R}^{2n}$ be $\Phi(z,w_1,\dots,w_{n-1})=(\xi,w_1,\dots,w_{n-1})$. Then $\Phi^*(g_0)=g$ where

$$g_0 = dr^2 + \beta^2 r^2 d\theta^2 + \sum_{i=1}^{n-1} |dw_i|^2$$

For f a function defined un a domain $U \subset \mathbb{C}^n$ and $0 < \alpha < 1$ we say that f belongs to $C^{,\alpha,\beta}$ if $f = \Phi^* \tilde{f}$ where $\tilde{f}$ is in C^α. Being in C^α means that

$$[f]_\alpha = \sup_{x,y} \frac{|f(x) - f(y)|}{|x-y|^\alpha} < \infty$$

Because $\beta^2 g_{euc} < g_0 < g_{euc}$ where g_{euc} is the euclidean metric on $\mathbb{R}^{2n}$ we can use either of them to define $[f]_\alpha$, in our case $|x-y|$ stands for the euclidean distance.

In $C^{,\alpha,\beta}$ we have the norm $\|f\|_{\alpha,\beta} = \|\tilde{f}\|_\alpha = sup_x|\tilde{f}(x)| + [\tilde{f}]_\alpha$

We say that a $(1,0)$ form is in $C^{,\alpha,\beta}$ if its components in $\{\eta, dw_i\}$ are in $C^{,\alpha,\beta}$, similarly a $(1,1)$ form is in $C^{,\alpha,\beta}$ if its components in $\{\eta \wedge \overline{\eta}, \eta \wedge d\overline{w_i}, dw_i \wedge \overline{\eta}, dw_i \wedge d\overline{w_j}\}$ are in $C^{,\alpha,\beta}$. In both of this spaces we have norms defined by summing the $\|\|_{\alpha,\beta}$ norms of the components.

Finally we set $C^{2,\alpha,\beta}$ to be the space of real functions f such that $f, \partial f, i\partial\overline{\partial} f$ are $C^{,\alpha,\beta}$ with the norm $\|f\|_{2,\alpha,\beta} = \|f\|_{\alpha,\beta} + \|\partial f\|_{\alpha,\beta} + \|i\partial\overline{\partial} f\|_{\alpha,\beta}$.

The basic fact we need is

Theorem 4 *Let $0 < \alpha < \beta^{-1} - 1$, B_1, B_2 the preimages under Φ of euclidean balls of radius 1 and 2. Then there exist a constant C depending only on α, β and n such that for every $f \in C^{2,\alpha,\beta}(B_2)$ we have*

$$\|f\|_{C^{2,\alpha,\beta}(B_1)} \leq C(\|\triangle_\Omega f\|_{C^{,\alpha,\beta}(B_2)} + \|f\|_{C^0(B_2)})$$

Let us work in $(\mathbb{R}^{2n}, g_0)$ with coordinates (r, θ, s_j) where $j = 1, \ldots, 2n-2$, $w_i = s_i + is_{n-1+i}$.

Consider the set of differential operators

$$S = \{\frac{\partial^2}{\partial s_j \partial s_k}, \frac{\partial^2}{\partial r \partial s_j}, \frac{1}{r}\frac{\partial^2}{\partial \theta \partial s_j}, \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{\beta^2 r^2}\frac{\partial^2}{\partial \theta^2}\}$$

Define the norm

$$\|f\|_* = \|f\|_{1,\alpha} + \sum_{D \in S} \|Df\|_\alpha$$

where $\|f\|_{1,\alpha} = \|f\|_0 + \|\nabla f\|_0 + [\nabla f]_\alpha$

The theorem follows from the estimate

$$\|f\|_{*(B_1)} \leq C(\|\triangle_{g_0} f\|_{C^\alpha(B_2)} + \|f\|_{C^0(B_2)})$$

If f is defined on $B(2R)$ (a ball of radius $2R$), then $f_R(x) = f(Rx)$ is defined on B_2. Applying the estimate to f_R we get

$$\|f\|_{C^0(B_R)} + R\|\nabla f\|_{C^0(B_R)} + R^{1+\alpha}[\nabla f]_{C^\alpha(B_R)} + \sum_{D \in S} R^2\|Df\|_{C^0(B_R)} + \sum_{D \in S} R^{2+\alpha}[Df]_{C^\alpha(B_R)} \leq$$

$$C(R^2\|\triangle_{g_0} f\|_{C^0(B_{2R})} + R^{2+\alpha}[\triangle_{g_0} f]_{C^\alpha(B_{2R})} + \|f\|_{C^0(B_{2R})})$$

Note that we used $\triangle_{g_0} f_R = R^2(\triangle_{g_0} f)_R$, this is not true for $\triangle_\Omega$.

Let $\rho \geq 1$ be a smooth weight function on $\mathbb{R}^{2n}$ with $\rho = r$ when $r \geq 2$. For $\gamma \in \mathbb{R}$ define C^0_γ to be the continuous functions f for which

$$\|f\|_{C^0_\gamma} = \sup_x \rho^{-\gamma}(x)|f(x)| < \infty$$

In other words $f = O(\rho^\gamma)$.

The subspace C^α_γ is composed by the functions for which

$$[f]_{C^\alpha_\gamma} = \sup_{x,y} \min(\rho(x), \rho(y))^{-\gamma+\alpha} \frac{|f(x) - f(y)|}{|x-y|^\alpha} < \infty$$

On C^α_γ we have the norm $\|f\|_{C^\alpha_\gamma} = \|f\|_{C^0_\gamma} + [f]_{C^\alpha_\gamma}$.

Motivated by the scaling behavior of the estimates, we define $\tilde{C}^{2,\alpha}_\gamma$ to de the space of $f \in C^\alpha_\gamma$ such that $\nabla f \in C^\alpha_{\gamma-1}$ and $Df \in C^\alpha_{\gamma-2}$ for every $D \in S$. We have the norm

$$\|f\|_{\tilde{C}^{2,\alpha}_\gamma} = \|f\|_{C^0_\gamma} + \|\nabla f\|_{C^\alpha_{\gamma-1}} + \sum_{D\in S} \|Df\|_{C^\alpha_{\gamma-2}}$$

Let $f \in C^\alpha_\gamma$ with $\gamma < -2$ and $u(x) = \int_{\mathbb{R}^{2n}} G(x,y)f(y)dy$. Then [1] shows that $u, \nabla u, Du$ are locally in C^α and $\triangle_{g_0} u = f$. We also have

Lemma 2 *If $-2n < \gamma < -2$ then $u \in C^0_{\gamma+2}$ and $\|u\|_{C^0_{\gamma+2}} \leq C\|f\|_{C^0_\gamma}$. The same conclusion holds if $-2n-1 < \gamma < -2n$ and $\int_{\mathbb{R}^{2n}} f = 0$*

Proof:

Let $m = 2n$ and $c = \beta^{-1}$. Because $|f(y)| \leq \|f\|_{C^0_\gamma}\rho(y)^\gamma$ it is enough to show that $\int_{\mathbb{R}^m} G(x,y)\rho(y)^\gamma dy$ is in $C^0_{\gamma+2}$. Split the integral in three regions: $|y| \geq 2|x|, |x|/2 \leq |y| \leq 2|x|, |y| \leq |x|/2$. You can assume that $|x| \geq 4$.

There exist a constant C such that $|G(w_1,z) - G(w_2,z)| \leq C|w_1 - w_2|^c$ and $G(0,z) \leq C$ for every z, w_1, w_2 with $|z| = 1, |w_1|, |w_2| \leq 1/2$. Using this and that $G(x,y) = \lambda^{2-n}G(\lambda^{-1}x, \lambda^{-1}y)$ for every $\lambda > 0$ we get that when $|y| \geq 2|x|$

$$G(x,y) = |y|^{2-n}G(|y|^{-1}x, |y|^{-1}y) \leq |y|^{2-n}(|G(|y|^{-1}x, |y|^{-1}y) - G(0, |y|^{-1}y)| + G(0, |y|^{-1}y))$$
$$\leq C|y|^{2-n}(|y|^{-c}|x|^c + 1) \leq C(1 + (1/2)^c)|y|^{2-n}$$

Then

$$\int_{|y|\geq 2|x|} G(x,y)\rho(y)^\gamma dy \leq C\int_{|y|\geq 2|x|} |y|^{2-n}|y|^\gamma dy = C\int_{2|x|}^\infty r^{2-n}r^\gamma r^{n-1}dr \leq C|x|^{\gamma+2}$$

When $|y| \geq |x|/2 \geq 2$ we have $|y|^\gamma \leq C|x|^\gamma$, we can also assume $\sup_{|z|=1}\int_{|y|\leq 2} G(y,z)dy \leq C$. Then

$$\int_{|x|/2\leq|y|\leq 2|x|} G(x,y)\rho(y)^\gamma dy \leq C|x|^\gamma \int_{|y|\leq 2|x|} G(x,y)dy = C|x|^\gamma\int_{|y|\leq 2} G(x,|x|y)|x|^n dy$$
$$= C|x|^\gamma\int_{|y|\leq 2} |x|^{2-n}G(|x|^{-1}x, y)|x|^n dy \leq C|x|^{\gamma+2}$$

When $|y| \leq |x|/2$, as in the first part, we have $G(x,y) \leq C|x|^{2-n}$ so

$$\int_{|y|\leq|x|/2} G(x,y)\rho(y)^\gamma dy = \int_{|y|\leq 2} G(x,y)\rho(y)^\gamma dy + \int_{2\leq|y|\leq|x|/2} G(x,y)|y|^\gamma dy$$
$$\int_{2\leq|y|\leq|x|/2} G(x,y)|y|^\gamma dy \leq C|x|^{2-n}\int_2^{|x|} r^\gamma r^{n-1}dr \leq C|x|^{\gamma+2}$$
$$\int_{|y|\leq 2} G(x,y)\rho(y)^\gamma dy \leq C|x|^{2-n} \leq C|x|^{\gamma+2} \qquad \text{if } -n < \gamma < -2$$

If $-n-1 < \gamma < -n$ and $\int f = 0$ then

$$u(x) = \int (G(x,y) - a\rho(x)^{2-n})f(y)dy$$

where $G(x,0) = a|x|^{2-n}$

Now $|G(x,y) - G(x,0)| \leq C|x|^{2-n}|x|^{-c}|y|^c$, so $\int_{|y|\leq 2} |G(x,y) - \rho(y)^{2-n}|dy \leq C|x|^{2-n-c} \leq C|x|^{1-n} \leq C|x|^{\gamma+2}$ if $-n-1 < \gamma < -n$

□

Let $x \in \mathbb{R}^n$ with $|x| \geq 4$, let B_{2R} be the ball centered at x with $2R = |x|/2$, the key point is that on B_{2R} we have $2R < \rho = r < 6R$. Multiplying the equation (1) by $R^{-\gamma-2}$ and using the lemma, we get

$$\|u\|_{\tilde{C}^{2,\alpha}_{\gamma+2}(B_R)} \leq C\|f\|_{C^{\alpha}_{\gamma}(\mathbb{R}^n)}$$

It is trivial to get an estimate $\|u\|_{\tilde{C}^{2,\alpha}_{\gamma+2}(B_4(0))} \leq C\|f\|_{C^{\alpha}_{\gamma}(\mathbb{R}^n)}$. Then it follows that

$$\|u\|_{C^0_\gamma} + \|\nabla u\|_{C^0_{\gamma-1}} + \sum_{D\in S} \|Du\|_{C^0_{\gamma-2}} \leq C\|f\|_{C^{\alpha}_{\gamma}}$$

We have to estimate the quantities $[\nabla u]_{C^{\alpha}_{\gamma-1}}$ and $[Du]_{C^{\alpha}_{\gamma-2}}$. It is enogh to consider two cases:

If $x \in B(0,4)$, $|y| \geq 4$, $x \notin B(y,|y|/4)$. Then $\min(\rho(x),\rho(y)) \leq 4$ and $|x-y| \geq 1$ so

$$\min(\rho(x),\rho(y))^{-\gamma+1+\alpha}\frac{|\nabla u(x)-\nabla u(y)|}{|x-y|^{\alpha}} \leq C\|\nabla u\|_{C^0}$$

and $\min(\rho(x),\rho(y))^{-\gamma+2+\alpha}\frac{|Du(x)-Du(y)|}{|x-y|^{\alpha}} \leq C\|Du\|_{C^0}$

If $|x|,|y| \geq 4$, $x \notin B(y,|y|/4)$ and $y \notin B(x,|x|/4)$ then $|x-y| \geq |x|/4$ and $|x-y| \geq |y|/4$ so

$$\min(\rho(x),\rho(y))^{-\gamma+1+\alpha}\frac{|\nabla u(x)-\nabla u(y)|}{|x-y|^{\alpha}} \leq$$

$$|x|^{-\gamma+1+\alpha}4^{\alpha}\frac{|\nabla u(x)|}{|x|^{\alpha}} + |y|^{-\gamma+1+\alpha}4^{\alpha}\frac{|\nabla u(y)|}{|y|^{\alpha}} \leq C\|\nabla u\|_{C^0_{\gamma-1}}$$

Similarly,

$$\min(\rho(x),\rho(y))^{-\gamma+2+\alpha}\frac{|Du(x)-Du(y)|}{|x-y|^{\alpha}} \leq$$

$$|x|^{-\gamma+2+\alpha}4^{\alpha}\frac{|Du(x)|}{|x|^{\alpha}} + |y|^{-\gamma+2+\alpha}4^{\alpha}\frac{|Du(y)|}{|y|^{\alpha}} \leq C\|Du\|_{C^0_{\gamma-2}}$$

So finally we get

$$\|u\|_{\tilde{C}^{2,\alpha}_{\gamma+2}(\mathbb{R}^n)} \leq C\|f\|_{C^{\alpha}_{\gamma}(\mathbb{R}^n)}$$

Going back to complex coordinates we define $C^{,\alpha,\beta}_{\gamma} = \Phi^*(C^{\alpha}_{\gamma})$ with $\|f\|_{C^{,\alpha,\beta}_{\gamma}} = \|\tilde{f}\|_{C^{,\alpha}_{\gamma}}$.

A (1, 0) form is in $C^{,\alpha,\beta}_{\gamma}$ if its components with respect to $\{\eta, dw_i\}$ are in $C^{,\alpha,\beta}_{\gamma}$. Similarly for (1, 1) forms using $\{\eta\wedge\overline{\eta}, \eta\wedge d\overline{w_i}, dw_i\wedge\overline{\eta}, dw_i\wedge d\overline{w_j}\}$. We have obvious norms by adding the norms of the components. Again let $C^{2,\alpha,\beta}_{\gamma}$ be the space of real functions such that $f \in C^{,\alpha,\beta}_{\gamma}$, $\partial f \in C^{,\alpha,\beta}_{\gamma-1}$ and $i\partial\overline{\partial} f \in C^{,\alpha,\beta}_{\gamma-2}$. We have the norm $\|f\|_{C^{2,\alpha,\beta}_{\gamma}} = \|\tilde{f}\|_{C^0_\gamma} + \|\partial f\|_{C^{,\alpha,\beta}_{\gamma-1}} + \|i\partial\overline{\partial} f\|_{C^{,\alpha,\beta}_{\gamma-2}}$

The main result of this section is the following

Proposition 4 *Let $f \in C^{,\alpha,\beta}_{\gamma}$ where $0 < \alpha < \beta^{-1}-1$, $-2n < \gamma < -2$. Then there exist a unique $u \in C^{2,\alpha,\beta}_{\gamma+2}$ such that $\triangle_\Omega u = f$. Moreover $\|u\|_{C^{2,\alpha,\beta}_{\gamma+2}} \leq C\|f\|_{C^{,\alpha,\beta}_{\gamma}}$. The same conclusion holds for $-2n-1 < \gamma < -2n$ if we add $\int_{\mathbb{C}^n} f\Omega^n = 0$*

C depends only on n, α, β, γ

References

[1] Donaldson, S. *Kahler metrics with cone singularities along a divisor*, arxiv (2011)

[2] Joyce, D. *Compact manifolds with special holonomy*, Oxford Mathematical Monograph, (2000).

[3] Le Brun, C. *Complete Ricci flat Kahler metrics on $\mathbb{C}^n$ need not be flat*, Proceedings of Symposia in Pure Mathematics, (1991)

[4] Troyanov, M. *Prescribing curvature on compact surfaces with conical singularities*, Transactions of the American Mathematical Society, (1991)